# LOWER ALGEBRAIC $K$-THEORY OF CERTAIN REFLECTION GROUPS.

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Abstract. For $P \subset \mathbb{H}^3$ a finite volume geodesic polyhedron, with the property that all interior angles between incident faces are of the form $\pi/m_{ij}$ ($m_{ij} \geq 2$ an integer), there is a naturally associated Coxeter group $\Gamma_P$. Furthermore, this Coxeter group is a lattice inside the semi-simple Lie group $O^+(3,1) = \mathrm{Isom}(\mathbb{H}^3)$, with fundamental domain the original polyhedron $P$. In this paper, we provide a procedure for computing the lower algebraic $K$-theory of the integral group ring of such groups $\Gamma_P$ in terms of the geometry of the polyhedron $P$. As an ingredient in the computation, we explicitly calculate the $K_{-1}$ and $Wh$ of the groups $D_n$ and $D_n \times \mathbb{Z}_2$, and we also summarize what is known about the $\tilde{K}_0$.

## 1. Introduction

Algebraic $K$-theory is a family of covariant functors from the category of rings to the category of abelian groups, and when applied to a ring $R$, yields information about the category of projective modules over $R$. The algebraic $K$-theory functors (and their relatives) are of great interest to topologists, particularly when applied to integral group rings of discrete groups. Indeed, it was discovered in the 1960's that, for various natural problems in geometric topology, obstructions to the solution (in high dimensions) appeared as all the members of suitable $K$-theory groups.

In view of this, one can understand the desire to obtain *explicit* computations for the $K$-theory of integral group rings of finitely generated groups. Recent work of Lafont-Ortiz [LO1], [LO2] gave explicit computations for the lower algebraic $K$-theory of all the *hyperbolic 3-simplex reflection groups*. These are the lattices in the Lie group $O^+(3,1) = \mathrm{Isom}(\mathbb{H}^3)$, that are generated by reflections in the sides of a suitable geodesic 3-simplex in $\mathbb{H}^3$. Such groups were classified by several different authors, and there are precisely 32 of them up to isomorphism (see the discussion in the introduction of [JKRT]).

Now consider $P \subset \mathbb{H}^3$ a finite volume geodesic polyhedron, with the property that all the interior angles between incident faces are of the form $\pi/m_{ij}$ ($m_{ij} \geq 2$ an integer). One can extend each of the (finitely many) faces to a hyperplane (i.e. totally geodesic $\mathbb{H}^2$ embedded in $\mathbb{H}^3$), and form the subgroup $\Gamma_P \leq O^+(3,1)$ generated by reflections in these hyperplanes. This group will always be a lattice in $O^+(3,1)$, with fundamental domain the original polyhedron $P$. We will call such a group a *3-dimensional hyperbolic reflection group*. A special case of this occurs if $P$ is a tetrahedron, in which case the group $\Gamma_P$ is one of the 32 hyperbolic 3-simplex reflection groups. This case of the tetrahedron is somewhat special, as for most other combinatorial types of *fixed* simple polytopes, there are in fact infinitely many distinct groups $\Gamma_P$ with polyhedron $P$ of the given combinatorial type (see for instance the Appendix for the case of the cube). In this paper, our goal is to explain how the methods of [LO2] can be extended to provide computations of the lower algebraic $K$-theory of the lattice $\Gamma_P \leq O^+(3,1)$ for arbitrary polyhedron $P$.

We present background material in Section 2. In particular, we remind the reader of existing results, that allow us to express the lower algebraic $K$-theory of the group $\Gamma_P$ (namely $Wh(\Gamma_P)$ for $*=1$, $\tilde{K}_0(\mathbb{Z}\Gamma_P)$ for $*=0$, and $K_*(\mathbb{Z}\Gamma_P)$ for $*<0$) as a direct sum:

$$H_*^{\Gamma_P}(E_{\mathcal{FIN}}(\Gamma_P); \mathbb{KZ}^{-\infty}) \oplus \bigoplus_{V \in \mathcal{V}} H_*^V(E_{\mathcal{FIN}}(V) \to *).$$

allowing us to break down its computation into that of the various summands.

The first term appearing in the above splitting is a suitable equivariant generalized homology group of a certain space. There is a spectral sequence which allows one to compute this homology group. The $E^2$-terms of this spectral sequence are obtained by taking the homology of a certain chain complex. The groups appearing in this chain complex are given by the lower algebraic $K$-theory of various finite groups, primarily dihedral groups $D_n$ and products $D_n \times \mathbb{Z}_2$. In Section 3, we provide explicit number theoretic formulas for the $K_{-1}$ and $Wh$ of these finite groups (Sections 3.1 and 3.4), and we summarize what is known about the $\tilde{K}_0$ of these groups (Section 3.3); the $K_*$ for $*<-1$ are well known to vanish.

In Section 4, we analyze the chain complex, compute the $E^2$-terms in the spectral sequence, and use this information to identify the term $H_*^{\Gamma_P}(E_{\mathcal{FIN}}(\Gamma_P); \mathbb{KZ}^{-\infty})$ within the range $* \leq 1$. For $*=0,1$, it is known that the remaining terms in the splitting are torsion. Specializing to the case $*=1$, this allows us (Theorem 4) to give an explicit formula for the rationalized Whitehead group $Wh(\Gamma_P)\otimes\mathbb{Q}$, in terms of the combinatorics and geometry of the polyhedron $P$. Similarly, for $* \leq -1$, the remaining terms in the splitting are known to vanish. This allows us (Theorem 5) to also provide similarly explicit, though much more complicated, expressions for $K_{-1}(\mathbb{Z}\Gamma_P)$.

In Section 5, we focus on identifying the remaining terms in the splitting. Using geometric techniques, we show that only finitely many of these terms are non-zero. Furthermore, we show that the non-vanishing terms consist of Bass Nil-groups, associated to various dihedral groups that can be identified from the geometry of the polyhedron $P$.

Overall, the results encompassed in this paper give a general procedure for computing the lower algebraic $K$-theory of any such $\Gamma_P$. More precisely, we can use the geometry of the polyhedron $P$ to:

- give a completely explicit computation for $K_{-1}(\mathbb{Z}\Gamma_P)$,
- give an expression for $\tilde{K}_0(\mathbb{Z}\Gamma_P)$ in terms of the $\tilde{K}_0$ of various dihedral groups, and products of dihedral groups with $\mathbb{Z}_2$ (the computation of which is a classical problem, see Section 4.2), as well as certain Bass Nil-groups $NK_0(\mathbb{Z}D_n)$,
- give an expression for $Wh(\Gamma_P)$ in terms of the Bass Nil-groups $NK_1(\mathbb{Z}D_n)$ associated to various dihedral groups (the calculation of which is also a well known, difficult problem).

In particular, we see that the lower algebraic $K$-theory of $\Gamma_P$ can be directly determined from the geometry of the polyhedron $P$.

Finally, in the Appendix to the paper (Section 6), we illustrate this process by computing the lower algebraic $K$-theory for a family of Coxeter groups $\Gamma_P$ whose associated polyhedron $P$ is the product of an $n$-gon with an interval (see Example 7). As a second class of examples, we compute the lower algebraic $K$-theory for an infinite family of Coxeter groups whose associated polyhedrons are combinatorial cubes (see Example 8).

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## 2. Background material: geodesic polyhedra in $\mathbb{H}^3$, Coxeter groups, and the FJIC

In this section, we introduce some background material. We briefly discuss the groups of interest (Section 2.1), and the possible polyhedra $P$ (Section 2.2). We also discuss how, given a Coxeter group $\Gamma$, we can algorithmically decide whether $\Gamma \cong \Gamma_P$ for some polyhedron $P \subset \mathbb{H}^3$ (Section 2.3). Finally, we provide a review of the Farrell-Jones isomorphism conjecture, and discuss its relevance to our problem (Section 2.4).

2.1. **Hyperbolic reflection groups.** Consider a geodesic polyhedron $P \subset \mathbb{H}^3$, having the property that every pair of incident faces intersects at an (internal) angle $\pi/m_{ij}$, where $m_{ij} \geq 2$ is an integer. Associated to such a polyhedron, one can form a labeled complete graph $\mathcal{G}$ as follows:

- associate a vertex $v_i$ to every face $F_i$ of $P$,
- if a pair of faces $F_i, F_j$ of $P$ intersect at an angle of $\pi/m_{ij}$, label the corresponding edge of $\mathcal{G}$ by the integer $m_{ij}$,
- if a pair of faces of $P$ do not intersect, label the corresponding edge of $\mathcal{G}$ by $\infty$.

From the resulting labeled graph $\mathcal{G}$, one can form a Coxeter group $\Gamma_P$ in the usual manner: one assigns a generator $x_i$ of order two to each vertex $v_i$ of $\mathcal{G}$, and adds in relations $(x_i x_j)^{m_{ij}} = 1$ to every labeled edge of $\mathcal{G}$ (with the understanding that a relation is vacuous if the exponent is $\infty$).

From the constraint on the angles of the polyhedron $P$, it is clear that one has a homomorphism $\Gamma_P \to O^+(3,1)$, obtained by assigning to each generator $x_i \in \Gamma_P$ the reflection in the hyperplane extending the corresponding face $F_i$ of $P$. This morphism is in fact an embedding of $\Gamma_P \hookrightarrow O^+(3,1)$ as a lattice, with fundamental domain precisely the original polyhedron $P$.

Conversely, we observe that, from the Coxeter graph $\mathcal{G}$, we can readily recover both the combinatorial polyhedron $P$ and the internal angles $\pi/m_{ij}$ associated to each edge in the 1-skeleton of $P$.

2.2. **Geodesic polyhedra in $\mathbb{H}^3$.** A natural question is the realization question: given a combinatorial polyhedron $P$ with prescribed internal angles of the form $\pi/m_e$ ($m_e \geq 2$) assigned to each edge $e$ of $P$, does it arise as a geodesic polyhedron inside $\mathbb{H}^3$? If so, we will say that the labeled combinatorial polyhedron is *realizable* in $\mathbb{H}^3$.

In fact, a celebrated result of Andreev [An] provides a complete characterization of finite volume geodesic polyhedra in $\mathbb{H}^3$ with non-obtuse internal angles. More precisely, given an abstract combinatorial polyhedron $P$, with a $0 < \theta_e \leq \pi/2$ assigned to each edge $e$, Andreev demonstrated that the following two statements are equivalent:

(1) the labeled combinatorial polyhedron $P$ is realizable in $\mathbb{H}^3$, and
(2) the collection $\theta_e$ satisfy a finite collection of linear inequalities, which are explicitly given in terms of the combinatorics of the polyhedron $P$.

For a more specific discussion, we refer the reader to the recent paper of Roeder, Hubbard, and Dunbar [RHD], or to the book of M. Davis [Da, Section 6.10]. The point we want to emphasize is that, from a combinatorial polyhedron with prescribed dihedral angles $\pi/m_e$, one can use Andreev's theorem to easily check whether the labeled combinatorial polyhedron is realizable in $\mathbb{H}^3$.

2.3. **Coxeter groups as hyperbolic reflection groups.** The Coxeter groups of interest here are canonically associated to a certain class of geodesic polyhedra $P$ in $\mathbb{H}^3$. For these Coxeter groups, our goal is to provide recipes for computing the lower algebraic $K$-theory. The reader might naturally be interested in knowing, given a Coxeter group, whether it is one of these 3-dimensional hyperbolic reflection groups. We summarize here the procedure for answering this question. Let us assume that we are given a Coxeter group $\Gamma$ in terms of its complete Coxeter graph $\mathcal{G}$ (i.e. the complete graph on the generators, with each edge labelled by either an integer $\geq 2$, or by $\infty$).

First of all, we note that if we exclude the edges labelled $\infty$ in the graph $\mathcal{G}$, we obtain a labeled graph $\mathcal{G}'$. If the Coxeter group $\Gamma$ was a hyperbolic reflection group, then $\mathcal{G}'$ would have to be the dual graph to the corresponding polyhedron $P$, and in particular, would have to be a *planar graph.* Furthermore, the fact that $\mathcal{G}'$ is dual to a polyhedron implies that it is 3-connected. So from now on, let us assume $\mathcal{G}'$ is a planar, 3-connected graph.

Note that by a famous result of Steinitz [St], 3-connected, planar graphs, are precisely the class of graphs that arise as 1-skeletons of polyhedra (in this case, the dual of $P$). Furthermore, a well-known result of Whitney [Wh] states that 3-connected planar graphs have a unique embedding in $S^2$. A detailed discussion of both these results can be found in Ziegler's classic text, see [Z, Ch. 4]. These two results now allow us to deal with $\mathcal{G}'$ as a polyhedron, since it arises as the 1-skeleton of a unique polyhedron.

Secondly, let us assume that $P$ is one of our geodesic polyhedra, and $\Gamma_P$ is the associated Coxeter group. If we take a vertex $v \in P$, then the stabilizer of $v$ under the $\Gamma_P$ has to be a 2-dimensional spherical reflection group. Since all the spherical Coxeter groups are in fact spherical triangle groups, this immediately implies that at most three faces of $P$ can contain $v$. So in particular, if $\mathcal{G}'$ is dual to the polyhedron $P$, we see that all the faces of $\mathcal{G}'$ corresponding to non-ideal vertices of $P$ must in fact be triangles.

Thirdly, if we take an *ideal vertex* $v$ of $P$, then the stabilizer of $v$ in $\Gamma_P$ must be a planar Coxeter group, generated by reflections in a planar polygon with all angles of the form $\pi/m_e$. Note that there are four such polygons: three triangles (equilateral triangle, right isosceles, and the $(\pi/6, \pi/3, \pi/2)$ triangle), or a square. This in turn implies that their are at most *four* faces in $P$ asymptotic to the vertex

$v$. So if $\mathcal{G}'$ is dual to $P$, then the faces of $\mathcal{G}'$ corresponding to ideal vertices of $P$ have either 3 or 4 sides. Putting this together we see that $\mathcal{G}'$ has the property that *all its faces are triangles or squares.*

Finally, given such an $\mathcal{G}'$, with labels $m_e$ on the edges (coming from the Coxeter graph), we can dualize $\mathcal{G}'$ to obtain a polyhedron $P$. We can assign to the edge $e^*$ of $P$ dual to the edge $e$ of $\mathcal{G}'$ the dihedral angle $\theta_e := \pi/m_e$. This gives a labeled combinatorial polyhedron, to which we can now apply Andreev's theorem, and find out whether it is realizable in $\mathbb{H}^3$. This allows us to efficiently determine whether the original, abstract Coxeter group $\Gamma$ is one to which the techniques of this paper apply.

Lastly, we point out that while these constraints are fairly stringent, this nevertheless allows for infinitely many pairwise non-isomorphic Coxeter groups $\Gamma_P$ (see for instance Example 8 in Section 6 for infinitely many examples with polyhedron a combinatorial cube).

2.4. **Isomorphism conjecture and splitting formulas.** The starting point for our computation is the Farrell-Jones isomorphism conjecture, which predicts, for a group $G$, that the natural map:

$$H_n^G(E_{\mathcal{VC}}G; \mathbb{KZ}^{-\infty}) \to H_n^G(*; \mathbb{KZ}^{-\infty}) \cong K_n(\mathbb{Z}G) \tag{1}$$

is an isomorphism for all $n$. This conjecture is known to hold for lattices in $O^+(3,1)$ for $n \leq 1$, by work of Farrell & Jones [FJ1] in the cocompact case, and by work of Berkove, Farrell, Juan-Pineda, and Pearson [BFJP] in the non-cocompact case. So to compute the (algebraic) right hand side, we can instead focus on computing the (topological) left hand side.

Let us discuss the left hand side. Farrell-Jones [FJ1] established the existence of an equivariant generalized homology theory (denoted $H_*^?(-; \mathbb{KZ}^{-\infty})$), having the property that for any group $G$, and any integer $n$, the equivariant homology of a point $*$ with trivial $G$-action satisfies $H_n^G(*; \mathbb{KZ}^{-\infty}) \cong K_n(\mathbb{Z}G)$. Now given any $G$-CW-complex $X$, the obvious $G$-equivariant map $X \to *$ induces a canonical homomorphism in equivariant homology:

$$H_n^G(X; \mathbb{KZ}^{-\infty}) \to H_n^G(*; \mathbb{KZ}^{-\infty}) \cong K_n(\mathbb{Z}G)$$

called the assembly map. The idea behind the isomorphism conjecture is to find a "suitable" space $X$ for the above map to be an isomorphism. The space $X$ should be canonically associated to the group $G$, and should be explicit enough for the left hand side to be computable. In the isomorphism conjecture, the space $E_{\mathcal{VC}}G$ that appears is any model for the classifying space for $G$-actions with isotropy in the family of virtually cyclic subgroups. We refer the reader to the survey paper by Lück and Reich [LR] for more details.

Having explained the left hand side of the Farrell-Jones isomorphism conjecture, let us now return to the groups for which we would like to do computations. Recall that we are considering groups $\Gamma_P$, associated to certain finite volume geodesic polyhedra $P \subset \mathbb{H}^3$. These groups are automatically lattices in $O^+(3,1)$.

Now for lattices $\Gamma \leq O^+(3,1)$, Lafont-Ortiz established in [LO2, Cor. 3.3] the following formula for the algebraic $K$-theory:

$$H_n^\Gamma(E_{\mathcal{VC}}(\Gamma); \mathbb{KZ}^{-\infty}) \cong H_n^\Gamma(E_{\mathcal{FIN}}(\Gamma); \mathbb{KZ}^{-\infty}) \oplus \bigoplus_{V \in \mathcal{V}} H_n^V(E_{\mathcal{FIN}}(V) \to *). \tag{2}$$

Let us explain the terms showing up in the above formula. The left hand side is the homology group we are interested in computing, and coincides with the algebraic $K$-theory groups $K_n(\mathbb{Z}\Gamma)$ (as equation (1) is an isomorphism). The first term on the right hand side is the equivariant homology of $E_{\mathcal{FIN}}\Gamma$, a model for the classifying space for proper actions of $G$. But it is well known that for lattices in $O^+(3,1)$, such a model is given by the action on $\mathbb{H}^3$. For the second term appearing on the right hand side, we have that:

- $\mathcal{V}$ consists of one representative $V$ from each conjugacy class in $\Gamma$ of those infinite subgroups of the form $\mathrm{Stab}_\Gamma(\gamma)$, where $\gamma$ ranges over geodesics in $\mathbb{H}^3$.
- the groups $H_n^V(E_{\mathcal{FIN}}(V) \to *)$ are the cokernels of the assembly maps $H_n^V(E_{\mathcal{FIN}}(V); \mathbb{KZ}^{-\infty}) \to H_n^V(*; \mathbb{KZ}^{-\infty})$.

In view of the splitting formula, we merely need to analyze the two terms appearing on the right hand side of equation (2).

The first term is computed via an Atiyah-Hirzebruch type spectral sequence, which we will analyze in Sections 3 and 4. The remaining terms will be analyzed in our last Section 5.

## 3. Lower algebraic $K$-theory of $D_n$, $D_n \times \mathbb{Z}_2$, and $A_5 \times \mathbb{Z}_2$

In order to compute the term $H_*^\Gamma(E_{\mathcal{FIN}}(\Gamma_P); \mathbb{KZ}^{-\infty})$, we will make use of an Atiyah-Hirzebruch type spectral sequence due to Quinn. The computation of the $E^2$-terms of the spectral sequence requires knowledge of the lower algebraic $K$-theory of cell stabilizers for the $\Gamma_P$-action on $\mathbb{H}^3$. For many of the groups arising as cell stabilizers, the lower algebraic $K$-theory is known (see [LO2, Section 5]). The only lower algebraic $K$-groups we still need to compute are those of dihedral groups $D_n$ (generic stabilizers of 1-cells), those of groups of the form $D_n \times \mathbb{Z}_2$ (generic stabilizers of 0-cells), as well as the $K_{-1}$ of the group $A_5 \times \mathbb{Z}_2$.

We recall that Carter [C] established that $K_n(\mathbb{Z}G) = 0$ for $n \leq -2$ whenever $G$ is a finite group. In particular, we will just focus on computing the $K_{-1}, \tilde{K}_0, Wh$ for the generic stabilizers $D_n$ and $D_n \times \mathbb{Z}_2$. Among these, we provide easily computable number theoretic expressions for the $K_{-1}$ (Section 3.1) and for $Wh$ (Section 3.4). In contrast, the determination of $\tilde{K}_0$ is a classical, hard question; we provide a summary of what is known (Section 3.3). We also calculate the group $K_{-1}(\mathbb{Z}[A_5 \times \mathbb{Z}_2])$ (Section 3.2).

### 3.1. The negative $K$-theory $K_{-1}(\mathbb{Z}G)$.

A general recipe for computing the $K_{-1}$ of integral group rings of finite groups is provided by Carter [C]. First recall that if $A$ is a simple artinian ring, it is isomorphic to $M_n(D)$ for some positive integer $n$ and division ring $D$, finite dimensional over its center $E$. That dimension $[D:E]$ is a square, and the *Schur index* of $A$ equals $\sqrt{[D:E]}$. For a field $F$ and a finite group $G$, let $r_F$ denote the number of isomorphism classes of simple $FG$-modules. D. W. Carter [C] proved

$$K_{-1}(\mathbb{Z}G) \cong \mathbb{Z}^r \oplus (\mathbb{Z}_2)^s$$

where

$$r = 1 - r_{\mathbb{Q}} + \sum_{p \mid |G|} (r_{\mathbb{Q}_p} - r_{\mathbb{F}_p}) \tag{3}$$

where, as in all our summations, $p$ is prime; and $s$ is the number of simple components $A$ of $\mathbb{Q}G$ with even Schur index but with $A_P$ of odd Schur index for each prime $P$ of the center of $A$ that divides $|G|$. We now proceed to use Carter's formula to compute the $K_{-1}$ associated to the groups $D_n, D_n \times \mathbb{Z}_2$, and $A_5 \times \mathbb{Z}_2$.

Let us first consider the case of the groups $D_n$ and $D_n \times \mathbb{Z}_2$. If $H$ is a subgroup of a group $G$, denote its index in $G$ by $[G : H]$. Suppose that $n$ is an integer exceeding 2, $\delta(n)$ is the number of (positive) divisors of $n$, and for each prime $p$, write $n = p^\nu \mu$ where $\mu \notin p\mathbb{Z}$. So $\nu = \nu_p(n)$ is the power to which $p$ divides $n$, and $\mu = \mu_p(n)$ is the non-$p$-part of $n$. Define

$$\sigma_p(n) = \sum_{d|\mu} [\, \mathbb{Z}_d^* : \langle -\bar{1}, \bar{p} \rangle \,] = \sum_{p \nmid d | n} [\, \mathbb{Z}_d^* : \langle -\bar{1}, \bar{p} \rangle \,]$$

$$\tau(n) = \sum_{p|2n} \nu_p(n) \sigma_p(n) = \sum_{p|n} \nu_p(n) \sigma_p(n)$$

We are now ready to state our:

**Theorem 1.** *If $D_n$ is the dihedral group of order $2n$ and $\mathbb{Z}_2$ is the cyclic group of order 2, then*

*(1)* $K_{-1}(\mathbb{Z}D_n) \cong \mathbb{Z}^{1-\delta(n)+\tau(n)}$,

*(2)* $K_{-1}(\mathbb{Z}[D_n \times \mathbb{Z}_2]) \cong \mathbb{Z}^{1-2\delta(n)+\sigma_2(n)+2\tau(n)}$.

*Proof.* For the groups $G = D_n$ and $G = D_n \times \mathbb{Z}_2$, we know that the simple components of $\mathbb{Q}G$ are matrix rings over fields (see below); so each has Schur index 1, which forces $s = 0$. This gives us the

**Fact 1:** For the groups $G = D_n$ and $G = D_n \times \mathbb{Z}_2$, the group $K_{-1}(\mathbb{Z}G)$ is torsion-free.

So for these groups, we are left with having to compute the quantities $r_F$ where $F$ are various fields. Now for $F$ a field of characteristic 0, $FG$ is semisimple, and $r_F$ coincides with the number of simple components of $FG$ in its Wedderburn decomposition.

Let us denote by $\epsilon$ the number of conjugacy classes of reflections in $D_n$; so $\epsilon$ is 1 or 2, according to whether $n$ is odd or even. As shown by Magurn [Ma1],

$$\mathbb{Q}D_n \cong \bigoplus_{d \,|\, n,\, d>2} M_2(\mathbb{Q}(\zeta_d + \zeta_d^{-1})) \oplus \mathbb{Q}^{2\epsilon};$$

so $r_{\mathbb{Q}} = \delta(n) + \epsilon$. On the other hand, for the group $D_n \times \mathbb{Z}_2$, we know that $\mathbb{Q}[D_n \times \mathbb{Z}_2] \cong \mathbb{Q}D_n \oplus \mathbb{Q}D_n$, which immediately tells us that for these groups, $r_{\mathbb{Q}} = 2\delta(n) + 2\epsilon$. We summarize these observations in our

**Fact 2:** For the groups $G = D_n$, we have that $r_{\mathbb{Q}} = \delta(n) + \epsilon$. For the groups $G = D_n \times \mathbb{Z}_2$, we have that $r_{\mathbb{Q}} = 2\delta(n) + 2\epsilon$.

To count simple $FG$-modules for an arbitrary field $F$ of characteristic $p$ (possibly $p = 0$), we employ a theorem of S. D. Berman. Suppose $d$ is a positive integer and $d \neq 0$ in $F$. Then $x^d - 1$ has $d$ different roots in the algebraic closure $\bar{F}$, and these form a (necessarily cyclic) subgroup of $\bar{F}^*$. Say $\zeta_d$ is a generator – a primitive $d^{\text{th}}$ root of unity over $F$. Now $F_d = F(\zeta_d)$ is a Galois extension of $F$, and each member of the Galois group $\mathrm{Aut}(F_d/F)$ is defined by its effect $\zeta_d \mapsto \zeta_d^t$, where $t$ is coprime to $d$. Sending such an automorphism to $\bar{t} \in \mathbb{Z}_d^*$ defines an embedding of the Galois group as a subgroup $T_d$ of $\mathbb{Z}_d^*$

With $p$ as above, an element $x \in G$ is *p-regular* if $p$ does not divide the order of $x$. Suppose $m$ is the least common multiple of the orders of all $p$-regular elements of $G$. Then $m \neq 0$ in $F$. Say $p$-regular elements $x, y \in G$ are *F-conjugate* if $x^t = gyg^{-1}$ for some $\bar{t} \in T_m$ and $g \in G$. This is an equivalence relation on the set of $p$-regular elements of $G$. Notice that $F$-conjugate $p$-regular elements have equal order, since each $t$ is coprime to $m$, hence to their orders. Berman established (see [CR1, Thms. (21.5) and (21.25)]) that for $F$ is field of characteristic $p$ (possibly $p = 0$) and $G$ a finite group, the number $r_F$ of isomorphism classes of simple $FG$-modules is equal to the number of $F$-conjugacy classes of $p$-regular elements of $G$. In view of this result and Carter's formula (3), we will now focus on counting the $F$-conjugacy classes in the groups $D_n$ and $D_n \times \mathbb{Z}_2$, where $F$ is either a $p$-adic field $\mathbb{Q}_p$ or a finite field $\mathbb{F}_p$. We first work over the field $\mathbb{Q}_p$, and establish

**Fact 3:** For the groups $G = D_n$, we have that $r_{\mathbb{Q}_p} = (\nu_p(n) + 1)\sigma_p(n) + \epsilon$. For the groups $G = D_n \times \mathbb{Z}_2$, we have that $r_{\mathbb{Q}_p} = 2(\nu_p(n) + 1)\sigma_p(n) + 2\epsilon$.

To see this, we first recall that for a prime $p$, the field $\mathbb{Q}_p$ of $p$-adic numbers has characteristic 0. Every element of a finite group $G$ is 0-regular, and hence the least common multiple $m$ of the orders of 0-regular elements coincides with the minimum exponent of $G$. We now set $F = \mathbb{Q}_p$, and for each $m$, denote by $F_m = \mathbb{Q}_p(\zeta_m)$. Write $m = p^e q$ with $e \geq 0$ and $q$ an integer not divisible by $p$.

As shown by [Se1, Chapter IV, Section 4], $\mathrm{Aut}(F_{p^e}/F)$ embeds as all $\mathbb{Z}_{p^e}^*$ and $\mathrm{Aut}(F_q/F)$ embeds as the cyclic subgroup $\langle \bar{p} \rangle$ in $\mathbb{Z}_q^*$. The former is deduced from

$$[F_{p^e} : F] = \phi(p^e) = |\mathbb{Z}_{p^e}^*|,$$

and this, in turn, follows from the irreducibility in $F[x]$ of the cyclotomic polynomial

$$p(x) = 1 + x^{p^{e-1}} + x^{2p^{e-1}} + \cdots + x^{(p-1)p^{e-1}}.$$

This irreducibility comes from that of the Eisenstein polynomial $p(x+1)$ in $F[x]$. Now $F_q$ is an unramified extension of $F$, so $p$ remains prime in its valuation ring and $p(x+1)$ is still an Eisenstein polynomial in $F_q[x]$. So $p(x)$ is irreducible there, and

$$[F_m : F_q] = [F_q(\zeta_{p^e}) : F_q] = \phi(p^e).$$

Therefore $[F_m : F] = [F_{p^e} : F][F_q : F]$.

Now $\zeta_{p^e}\zeta_q$ has order $m$. So we can choose $\zeta_m$ to be $\zeta_{p^e}\zeta_q$, and each element of $\mathrm{Aut}(F_m/F)$ is uniquely determined by its restrictions to $F_{p^e}$ and $F_q$. The resulting embedding

$$\mathrm{Aut}(F_m/F) \longrightarrow \mathrm{Aut}(F_{p^e}/F) \times \mathrm{Aut}(F_q/F)$$

must be an isomorphism, since the domain and codomain have equal finite size. Note also that, for $d$ dividing $m$, the restriction map

$$\mathrm{Aut}(F_m/F) \longrightarrow \mathrm{Aut}(F_d/F)$$

is surjective by the Extension Theorem of Galois theory. So the canonical map $\mathbb{Z}_m \to \mathbb{Z}_d$ takes $T_m$ onto $T_d$.

Next, let us specialize to $G = D_n$, with $F$ still the $p$-adic field $\mathbb{Q}_p$. Then $m$ is the least common multiple of 2 and $n$. For $\bar{t} \in T_m \leq \mathbb{Z}_m^*$, $t$ is coprime to $m$, so must be odd. This implies that for each reflection $b \in D_n$, $b^t = b$, telling us that the

$\mathbb{Q}_p$-conjugacy class of $b$ coincides with its ordinary conjugacy class. Hence there are $\epsilon$ distinct $\mathbb{Q}_p$-conjugacy classes of reflections in $D_n$.

Now let us consider rotations in $D_n$. Each rotation $x \in D_n$ has order dividing $n = p^\nu \mu$, and hence has order $p^i d$ with $i \leq \nu$ and $d$ dividing $\mu$. Every rotation of order $p^i d$ is uniquely expressible as a product $yz$ where $y$ and $z$ are rotations of orders $p^i$, $d$ respectively. If $x$ has this decomposition $yz$, then as $\bar{t}$ varies in $T_m$, $x^t = y^t z^t = y^j z^k$ where $(\bar{j}, \bar{k})$ runs through all pairs in $T_{p^i} \times T_d$. Since rotations $x \in D_n$ have conjugacy class $\{x, x^{-1}\}$, the $\mathbb{Q}_p-$conjugacy class of a rotation $x$ of order $p^i d$ is the set of

$$|T_{p^i} \times T_d| = |\mathbb{Z}^*_{p^i} \times \langle \bar{p} \rangle| = \phi(p^i)|\langle \bar{p} \rangle|$$

elements $x^t$, together with their inverses.

If $\langle \bar{p} \rangle = \langle -\bar{1}, \bar{p} \rangle$ in $\mathbb{Z}^*_d$, each $-\bar{p}^u$ is $p^v$ for some integer $v$, and the set of powers

$$x^t = y^u z^{p^u}$$

is closed under inverses. Or if $\langle \bar{p} \rangle \neq \langle -\bar{1}, \bar{p} \rangle$, then $-\bar{p}^u \notin \langle \bar{p} \rangle$, and the set of powers $x^t$ does not overlap the set of $x^{-t}$. Either way the number of $\mathbb{Q}_p$-conjugacy classes of $x$ is $\phi(p^i)|\langle -\bar{1}, \bar{p} \rangle|$. There are $\phi(p^i)\phi(d)$ rotations of order $p^i d$; so the number of $\mathbb{Q}_p$-conjugacy classes of them is the index

$$\frac{\phi(d)}{|\langle -\bar{1}, \bar{p} \rangle|} = [\mathbb{Z}^*_d : \langle -\bar{1}, \bar{p} \rangle].$$

This number is independent of $i$. Since there are $\nu + 1$ powers $p^i$, we obtain

$$r_{\mathbb{Q}_p} = (\nu + 1) \sum_{d|\mu} [\mathbb{Z}^*_d : \langle -\bar{1}, \bar{p} \rangle] + \epsilon = (\nu + 1)\sigma_p(n) + \epsilon$$

Finally, we observe that we have an isomorphism $\mathbb{Q}_p[D_n \times \mathbb{Z}_2] \cong \mathbb{Q}_p D_n \oplus \mathbb{Q}_p D_n$. Recalling that the integer $r_{\mathbb{Q}_p}$ can also be interpreted as the number of simple components in the Wedderburn decomposition of $\mathbb{Q}_p G$, this tells us that for the group $D_n \times \mathbb{Z}_2$, we have $r_{\mathbb{Q}_p} = 2(\nu_p(n)+1)\sigma_p(n)+2\epsilon$. This concludes the verification of Fact 3.

Finally, we work over the finite fields $\mathbb{F}_p$, and establish

**Fact 4:** For the groups $G = D_n$, we have that $r_{\mathbb{F}_p} = \sigma_p(n) + \epsilon$ ($p$ odd) and $r_{\mathbb{F}_2} = \sigma_2(n)$. For the groups $G = D_n \times \mathbb{Z}_2$, we have that $r_{\mathbb{F}_p} = 2\sigma_p(n) + 2\epsilon$ ($p$ odd) and $r_{\mathbb{F}_2} = \sigma_2(n)$.

Let us first consider the case $G = D_n$. Set $F = \mathbb{F}_p$, and let $m$ be the least common multiple of the orders of the $p$-regular elements in $G$. The fields $F$, $F(\zeta_m)$ are finite and $\mathrm{Aut}(F(\zeta_m)/F)$ is cyclic, generated by the $p$-power map, since $|F| = p$. So $T_m = \langle \bar{p} \rangle \leq \mathbb{Z}^*_m$. Fix a reference rotation $a \in D_n$ of order $n$.

The $p$-regular rotations are the rotations of order $d$ dividing $\mu_p(n)$, and so are of the form $\alpha^u$, where $\alpha = a^{n/d}$ and $u$ is coprime to $d$. Then $\alpha^u$, $\alpha^v$ are $\mathbb{F}_p$-conjugate if and only if

$$\alpha^{up^i} = \alpha^{v(-1)^j}$$

for some positive integers $i$, $j$. This just means $\bar{u} \equiv \bar{v} \bmod \langle -\bar{1}, \bar{p}\rangle$ in $\mathbb{Z}_d^*$. So the number of $\mathbb{F}_p$-conjugacy classes of $p$-regular rotations in $D_n$ is

$$\sum_{d|\mu} [\mathbb{Z}_d^* : \langle -\bar{1}, \bar{p}\rangle] = \sigma_p(n).$$

Next let us consider reflections in $D_n$. If $p$ is an odd prime, reflections are also $p$-regular. Since $m$ is even, each $t$ with $\bar{t} \in T_m \leq \mathbb{Z}_m^*$ is odd. So we see that for reflections, $\mathbb{F}_p$-conjugacy coincides with ordinary conjugacy. In particular, for odd primes, we see that the number of $\mathbb{F}_p$-conjugacy classes of reflections is $\epsilon$. On the other hand, reflections are not 2-regular. Putting this together, we obtain that

$$r_{\mathbb{F}_p} = \sigma_p(n) + \epsilon \quad (p \text{ odd}), \text{ and } r_{\mathbb{F}_2} = \sigma_2(n).$$

which establishes the first part of Fact 4.

Now let us consider groups of the form $G = D_n \times \mathbb{Z}_2$, where $\mathbb{Z}_2$ is the cyclic group $\{1, c\}$ generated by $c$ or order 2. In the group $D_n \times \mathbb{Z}_2$, the subgroup $D_n \times \{1\}$ is a copy of $D_n$. Fix a reference rotation $a \in D_n$ of order $n$ and a reference reflection $b \in D_n$. Then in $D_n \times \mathbb{Z}_2$ we have rotations $(a^i, 1)$, reflections $(a^i b, 1)$, co-rotations $(a^i, c)$ and co-reflections $(a^i b, c)$. For $x \in D_n$, the conjugates in $D_n \times \mathbb{Z}_2$ of $(x, 1)$ (resp. $(x, c)$) are the elements $(y, 1)$ (resp. $(y, c)$) with $x$ conjugate to $y$ in $D_n$. The least common multiple $m$ of orders of $p$-regular elements is the same for $D_n$ and $D_n \times \mathbb{Z}_2$; so the exponents $t$ with $\bar{t} \in T_m$ are the same for both these groups.

For $p$ odd, we have that $m$ is even, hence $t$ are odd, and $(x,1)^t = (x^t, 1)$, while $(x,c)^t = (x^t, c)$. The $p$-regular elements in $D_n \times \mathbb{Z}_2$ are the reflections, the co-reflections, and the $p$-regular rotations and co-rotations. So for $p$ odd, the number of $\mathbb{F}_p$-conjugacy classes in $D_n \times \mathbb{Z}_2$ is double the number of $\mathbb{F}_p$-conjugacy classes in the corresponding $D_n$:

$$r_{\mathbb{F}_p} = 2\sigma_p(n) + 2\epsilon \quad (p \text{ odd}).$$

On the other hand, the only 2-regular elements in $D_n \times \mathbb{Z}_2$ are rotations. This implies that the $\mathbb{F}_2$-conjugacy classes are the same for $D_n \times \mathbb{Z}_2$ as for $D_n$, giving us $r_{\mathbb{F}_2} = \sigma_2(n)$. This completes the verification of the second statement in Fact 4.

Finally, let us apply Carter's formula to complete the proof of our theorem. For groups of the form $G = D_n$, we have from Fact 1 that $K_{-1}(\mathbb{Z}D_n)$ is free abelian. Furthermore, the rank of $K_{-1}(\mathbb{Z}D_n)$ is given by equation (3). Combining Fact 3 and Fact 4, we see that

$$r_{\mathbb{Q}_p} - r_{\mathbb{F}_p} = \begin{cases} \nu_p(n)\sigma_p(n), & p \neq 2 \\ \nu_2(n)\sigma_2(n) + \epsilon, & p = 2. \end{cases}$$

Summing over all primes dividing $|G| = 2n$, we see that:

$$\sum_{p|2n} (r_{\mathbb{Q}_p} - r_{\mathbb{F}_p}) = \left( \sum_{p|2n} \nu_p(n)\sigma_p(n) \right) + \epsilon = \tau(n) + \epsilon.$$

Substituting this expression into equation (3), and substituting in the calculation of $r_{\mathbb{Q}}$ from our Fact 2, we see that the rank of $K_{-1}(\mathbb{Z}D_n)$ is given by:

$$r = 1 - (\delta(n) + \epsilon) - (\tau(n) + \epsilon) = 1 - \delta(n) + \tau(n),$$

which establishes part (1) of our Theorem 1.

Similarly, for groups of the form $G = D_n \times \mathbb{Z}_2$, we again have from Fact 1 that $K_{-1}(\mathbb{Z}D_n)$ is free abelian. Combining Fact 3 and Fact 4, we see that for these groups

$$r_{\mathbb{Q}_p} - r_{\mathbb{F}_p} = \begin{cases} 2\nu_p(n)\sigma_p(n), & p \neq 2 \\ 2\nu_2(n)\sigma_2(n) + \sigma_2(n) + 2\epsilon, & p = 2. \end{cases}$$

Summing over all primes dividing $|G| = 4n$ (which coincides with the primes dividing $2n$), we see that:

$$\sum_{p|4n}(r_{\mathbb{Q}_p} - r_{\mathbb{F}_p}) = 2\left(\sum_{p|2n}\nu_p(n)\sigma_p(n)\right) + \sigma_2(n) + 2\epsilon = 2\tau(n) + \sigma_2(n) + 2\epsilon.$$

Finally, substituting these into equation (3), and substituting the calculation of $r_{\mathbb{Q}}$ from our Fact 2, we obtain that $K_{-1}(\mathbb{Z}[D_n \times \mathbb{Z}_2])$ has rank

$$r = 1 - (2\delta(n) + 2\epsilon) + (2\tau(n) + \sigma_2(n) + 2\epsilon) = 1 - 2\delta(n) + \sigma_2(n) + 2\tau(n),$$

establishing part (2) of our theorem. This concludes the proof of Theorem 1. □

As an application of this result, we can easily determine the $K_{-1}$ of dihedral groups $D_n$ when $n$ has few divisors. For example, we have:

**Corollary 2.** *For $p$ an odd prime, set $r = [\mathbb{Z}_p^* : \langle -\bar{1}, \bar{2}\rangle]$. Then we have*

- $K_{-1}(\mathbb{Z}D_{2p}) \cong K_{-1}(\mathbb{Z}[D_p \times \mathbb{Z}_2]) \cong \mathbb{Z}^r$,
- $K_{-1}(\mathbb{Z}[D_{2p} \times \mathbb{Z}_2]) \cong \mathbb{Z}^{3r}$.

*Proof.* From our theorem, we have that the rank of $K_{-1}(\mathbb{Z}D_{2p})$ is given by:

$$1 - \delta(2p) + \tau(2p).$$

We have that $\delta(2p) = 4$, while $\tau(2p) = \nu_2(2p)\sigma_2(2p) + \nu_p(2p)\sigma_p(2p)$. Since $\nu_2(2p) = \nu_p(2p) = 1$, we are left with computing the integers $\sigma_2(2p), \sigma_p(2p)$. But it is easy to verify that $\sigma_p(2p) = 2$, while $\sigma_2(2p) = 1 + [\mathbb{Z}_p^* : \langle -\bar{1}, \bar{2}\rangle] = 1 + r$. This gives us $\tau(2p) = 3 + r$; substituting in these values gives the first statement in our corollary.

Similarly, we know that the rank of $K_{-1}(\mathbb{Z}[D_{2p} \times \mathbb{Z}_2])$ is given by:

$$1 - 2\delta(2p) + \sigma_2(2p) + 2\tau(2p).$$

Substituting in the values computed above, we find that the rank is given by

$$1 - 2(4) + (1 + r) + 2(3 + r) = 3r$$

which establishes the second statement in the corollary. □

For a concrete example, we see for instance that when $p = 3$, $r = 1$, then $K_{-1}(\mathbb{Z}D_6) \cong \mathbb{Z}$ and $K_{-1}(\mathbb{Z}[D_6 \times \mathbb{Z}_2]) \cong \mathbb{Z}^3$ (see [LO2, Section 5.1] for an alternate computation of this group).

As another application, we can completely classify dihedral groups (and products of dihedral groups with $\mathbb{Z}_2$) whose $K_{-1}$ vanishes:

**Corollary 3.** *$K_{-1}(\mathbb{Z}D_n) = 0$ if and only if $n$ is a prime power, and $K_{-1}(\mathbb{Z}[D_n \times \mathbb{Z}_2]) = 0$ if and only if $n$ is a power of 2.*

*Proof.* Note that

$$\sigma_p(n) = \sum_{d|\mu}[\mathbb{Z}_d^* : \langle -\bar{1}, \bar{p}\rangle] \geq \sum_{d|\mu} 1 = \delta(\mu) = \frac{\delta(n)}{\delta(p^\nu)} = \frac{\delta(n)}{1 + \nu}.$$

So

$$\tau(n) \geq \sum_{p|n} \nu_p(n)\left(\frac{\delta(n)}{1+\nu_p(n)}\right) = \delta(n)\sum_{p|n}\left(1-\frac{1}{1+\nu_p(n)}\right) \geq \delta(n)\sum_{p|n}\frac{1}{2} = \delta(n)\frac{g}{2},$$

where $g$ is the number of primes that divide $n$. Then the rank of $K_{-1}(\mathbb{Z}D_n)$ is

$$1-\delta(n)+\tau(n) \geq 1+\delta(n)\Big(\frac{g}{2}-1\Big).$$

So $K_{-1}(\mathbb{Z}D_n)$ does not vanish if $n$ is not a prime power. If $n = p^e$ for $p$ prime and $e \geq 1$, then $\tau(n) = \nu_p(n)\sigma_p(n) = (e)(1) = e$ and $\delta(n) = e+1$, which gives us $K_{-1}(\mathbb{Z}D_{p^e}) = 0$.

Next we consider the case of groups $D_n \times \mathbb{Z}_2$. For arbitrary $n$,

$$\sigma_2(n) + 2\tau(n) \geq \delta(n)\left[\frac{1}{1+\nu_2(n)} + g\right] > \delta(n)g.$$

This allows us to estimate from below the rank of $K_{-1}(\mathbb{Z}[D_n \times \mathbb{Z}_2]) =$

$$1-2\delta(n)+\sigma_2(n)+2\tau(n) > 1+\delta(n)(g-2).$$

So $K_{-1}(\mathbb{Z}[D_n \times \mathbb{Z}_2])$ doesn't vanish if $n$ is not a prime power. If $n = p^e$ for $p$ an odd prime and $e \geq 1$, its rank is given by

$$1-2(e+1)+\sigma_2(n)+2e = \sigma_2(n)-1.$$

But for such an $n$, $\nu_2(n) = 0$ and $\delta(n) = 1+e$, giving us

$$\sigma_2(n) \geq \frac{\delta(n)}{1+\nu_2(n)} = e+1 \geq 2.$$

This implies that $K_{-1}(\mathbb{Z}[D_{p^e} \times \mathbb{Z}_2])$ does not vanish. However, if $n = 2^e$ with $e \geq 1$, the rank is

$$1-2(e+1)+1+2e = 0$$

concluding the proof of Corollary 3. □

3.2. **The lower $K$-group $K_{-1}(\mathbb{Z}[A_5 \times \mathbb{Z}_2])$.** Recall that the group $A_5 \times \mathbb{Z}_2$ is one of the three "exceptional" groups that arise as stabilizers of vertices of $P$, along with $S_4$ and $S_4 \times \mathbb{Z}_2$. The lower $K$-groups of the last two groups have all been previously computed (see [LO2, Section 5]). We implement the method discussed in the previous section to show the following:

**Example 4.** $K_{-1}(\mathbb{Z}[A_5 \times \mathbb{Z}_2]) \cong \mathbb{Z}^2$.

*Proof.* We first recall that the group algebra $\mathbb{Q}A_5$ decomposes into simple components as follows:

$$\mathbb{Q}A_5 \cong \mathbb{Q} \oplus M_3(\mathbb{Q}(\sqrt{5})) \oplus M_4(\mathbb{Q}) \oplus M_5(\mathbb{Q}).$$

Since $\mathbb{Q}[A_5 \times \mathbb{Z}_2] \cong \mathbb{Q}A_5 \oplus \mathbb{Q}A_5$, we see that the Schur indices of all the simple components in the Wedderburn decomposition of $\mathbb{Q}[A_5 \times \mathbb{Z}_2]$ are equal to 1. Carter's result [C] now tells us that $K_{-1}(\mathbb{Z}[A_5 \times \mathbb{Z}_2])$ is torsion-free, and from equation (3), the rank is given by

$$r = 1 - r_{\mathbb{Q}} + (r_{\mathbb{Q}_2} - r_{\mathbb{F}_2}) + (r_{\mathbb{Q}_3} - r_{\mathbb{F}_3}) + (r_{\mathbb{Q}_5} - r_{\mathbb{F}_5}). \tag{4}$$

We now proceed to compute the various terms appearing in the above expression.

Recall that for $F$ a field of characteristic 0, $r_F$ just counts the number of simple components in the Wedderburn decomposition of the group algebra $F[A_5 \times \mathbb{Z}_2]$. From the discussion in the previous paragraph, we have that

$$\mathbb{Q}[A_5 \times \mathbb{Z}_2] \cong \mathbb{Q}^2 \oplus M_3\big(\mathbb{Q}(\sqrt{5})\big)^2 \oplus M_4(\mathbb{Q})^2 \oplus M_5(\mathbb{Q})^2.$$

yielding $r_{\mathbb{Q}} = 8$. Now by tensoring the above splitting with $\mathbb{Q}_p$, we obtain:

$$\mathbb{Q}_p[A_5 \times \mathbb{Z}_2] \cong \mathbb{Q}_p^2 \oplus M_3\big(\mathbb{Q}_p \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{5})\big)^2 \oplus M_4(\mathbb{Q}_p)^2 \oplus M_5(\mathbb{Q}_p)^2.$$

The second term is isomorphic to $M_3(\mathbb{Q}_p(\sqrt{5}))^2$ for $p = 2, 3$ and 5, since 5 is not a square mod 8, 3, or 25, hence not a square in $\mathbb{Q}_p$. In particular, for each of the primes $p = 2, 3, 5$, we obtain that $r_{\mathbb{Q}_2} = r_{\mathbb{Q}_3} = r_{\mathbb{Q}_5} = 8$.

Next let us consider the situation over the finite fields $\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_5$. We first recall that the integer $r_{\mathbb{F}_p}$ counts the number of $\mathbb{F}_p$-conjugacy classes of $p$-regular elements. Recall that an $\mathbb{F}_p$-conjugacy class of an element $x$ is the union of ordinary conjugacy classes of certain specific powers of $x$, where the powers are calculated from the Galois group associated to a finite extension of the field $\mathbb{F}_p$.

For $p = 2$, we note that elements in $A_5 \times \mathbb{Z}_2$ are 2-regular precisely if they have order 1, 3 or 5. There is a single conjugacy class of elements of order one (the identity element). The elements of order 3 form a single conjugacy class inside $A_5 \times \mathbb{Z}_2$. Finally, the elements of order 5 form *two* conjugacy classes in $A_5 \times \mathbb{Z}_2$; representatives for these two conjugacy classes are given by $g = \big((abcde), 1\big)$, and by $g^2$. So there will be either one or two $\mathbb{F}_2$-conjugacy classes of elements of order 5. To determine the specific powers, we note that the minimal exponent of $A_5 \times \mathbb{Z}_2$ equals $30 = 2 \cdot 15$. The powers of $x$ are given by considering the Galois group of the extension $Gal\big(\mathbb{F}_2(\zeta_{15})/\mathbb{F}_2\big)$, viewed as elements of $\mathbb{Z}_{15}^*$. Since the Galois group is generated by squaring, we see that the Galois group is cyclic of order 4, given by the residue classes $\{\bar{1}, \bar{2}, \bar{4}, \bar{8}\} \subset \mathbb{Z}_{15}^*$. In particular, since $\bar{2}$ lies in the Galois group, we see that $g$ and $g^2$ lie in the same $\mathbb{F}_2$-conjugacy class, implying that there is a *unique* $\mathbb{F}_2$-conjugacy class of elements of order 5. We conclude that there are three $\mathbb{F}_2$-conjugacy classes of 2-regular elements, giving $r_{\mathbb{F}_2} = 3$.

For $p = 3$, the elements in $A_5 \times \mathbb{Z}_2$ which are 3-regular have order 1, 2, 5, or 10. Since the minimal exponent of the group is $30 = 3 \cdot 10$, we look at the Galois group associated to the field extension $\mathbb{F}_3(\zeta_{10})$. Elements in the Galois group are generated by the third power, giving us that $Gal(\mathbb{F}_3(\zeta_{10})/\mathbb{F}_3) = \{\bar{1}, \bar{3}, \bar{7}, \bar{9}\} \subset \mathbb{Z}_{10}^*$. In particular, the $\mathbb{F}_3$-conjugacy class of any element $x \in A_5 \times \mathbb{Z}_2$ is the union of the conjugacy classes of the elements $x, x^3, x^7$, and $x^9$. Now we clearly have a unique $\mathbb{F}_3$-conjugacy class of elements of order one. For elements of order 2, there are *three* distinct (ordinary) conjugacy classes of elements of order two; each of these ordinary conjugacy class is also an $\mathbb{F}_3$-conjugacy class. Next we note that there are *two* conjugacy classes of elements of order 5, but these two ordinary conjugacy classes are part of a single $\mathbb{F}_3$-conjugacy class since the $7^{th}$ power of such an element is its square. Finally, we observe that there are *two* conjugacy classes of elements of order 10, but these again give rise to a single $\mathbb{F}_3$-conjugacy class. We conclude that overall there are six $\mathbb{F}_3$-conjugacy classes of 3-regular elements, giving $r_{\mathbb{F}_3} = 6$.

Finally, for $p = 5$, the elements in $A_5 \times \mathbb{Z}_2$ which are 5-regular have order 1, 2, 3, or 6. Since the minimal exponent of the group is $30 = 5 \cdot 6$, we need to look at the Galois group associated to the field extension $\mathbb{F}_5(\zeta_6)$. Elements in the Galois group are generated by the fifth power, giving us that $Gal(\mathbb{F}_5(\zeta_6)/\mathbb{F}_5) = \{\bar{1}, \bar{5}\} \subset \mathbb{Z}_6^*$.

This yields that the $\mathbb{F}_5$-conjugacy class of an element $x \in A_5 \times \mathbb{Z}_2$ of order five is the union of the ordinary conjugacy classes of $x$ and of $x^5$. Now we have a single $\mathbb{F}_5$-conjugacy class of elements of order one. For elements of order 2, we have *three* ordinary conjugacy classes of elements; but each of these also forms a single $\mathbb{F}_5$-conjugacy class (since for these elements, $x^5 = x$). For elements of order 3, we have a single ordinary conjugacy class of such elements, which also form a single $\mathbb{F}_5$-conjugacy class. Finally, for elements of order 6, we also have a single ordinary conjugacy class of such elements, which hence also form a single $\mathbb{F}_5$-conjugacy class. We conclude that there is a total of six $\mathbb{F}_5$-conjugacy classes of 5-regular elements, and hence $r_{\mathbb{F}_5} = 6$.

To conclude, we substitute in our calculations into the expression in equation (4) for the rank of $K_{-1}(\mathbb{Z}[A_5 \times \mathbb{Z}_2])$, giving us:

$$r = 1 - 8 + (8 - 3) + (8 - 6) + (8 - 6) = 2$$

completing our calculation for Example 4. □

3.3. **The class group $\tilde{K}_0(\mathbb{Z}G)$.** The group $\tilde{K}_0(\mathbb{Z}G) = K_0(\mathbb{Z}G)/\langle[\mathbb{Z}G]\rangle$ is closely related to the ideal class group of the ring of algebraic integers $R$ in a number field $F$ (that is, a field $F$ with $[F : \mathbb{Q}]$ finite).

The *ideal class group* $Cl(R)$ is the group of $R$-linear isomorphism classes of non-zero ideals of $R$, under multiplication of ideals: $(I)(J) = (IJ)$. The identity element is the class $(R)$ of non-zero principal ideals; so $Cl(R)$ measures the deviation of $R$ from being a principal ideal domain.

Suppose $A$ is a finite dimensional $\mathbb{Q}$-algebra. A *$\mathbb{Z}$-order* in $A$ is a subring $\Lambda$ of $A$ that is finitely generated as a $\mathbb{Z}$-module and spans $A$ as a $\mathbb{Q}$-vector space. For each prime number $p$, the set $\mathbb{Z} - p\mathbb{Z}$ is a submonoid of $\mathbb{Z}$ under multiplication; the local ring

$$S^{-1}\mathbb{Z} = \left\{ \frac{a}{n} : a \in \mathbb{Z}, n \in S \right\}$$

is denoted by $\mathbb{Z}_{(p)}$. For any $\Lambda$-module $M$, the localization $M_{(p)} = S^{-1}M$ is a $\Lambda_{(p)} = S^{-1}\Lambda$-module. We say $M$ is *locally free* if there is an integer $n \geq 0$ so that, for all primes $p$, $M_{(p)} = \Lambda^n_{(p)}$ as a $\Lambda_{(p)}$-modules. This $n$ is the rank $rk(M)$ of $M$. By [Sw, Lemma 6.14], finitely generated locally free $\Lambda$-modules are projective; so they generate a subgroup $LF(\Lambda)$ of $K_0(\Lambda)$. There is a surjective group homomorphism

$$rk : LF(\Lambda) \to \mathbb{Z}, \quad [P] - [Q] \mapsto rk(P) - rk(Q).$$

Its kernel is the *locally free class group* $Cl(\Lambda)$ of the $\mathbb{Z}$-order $\Lambda$. Since $rk$ is split by sending $1 \in \mathbb{Z}$ to $[\Lambda] \in LF(\Lambda)$, there is an isomorphism $Cl(\Lambda) \cong LF(\Lambda)/\langle[\Lambda]\rangle$. As shown in [CR2, 39.13], $Cl(\Lambda)$ is a finite group.

In the classical case where $A$ is a number field $F$ and $\Lambda$ is its ring of algebraic integers $R$, $Cl(R)$ as defined above coincides with the classical ideal class group $Cl(R)$, and with $\tilde{K}_0(R)$. Its order $h(F)$ is called the *class number* of $F$.

Our focus is the case $A = \mathbb{Q}G$, $\Lambda = \mathbb{Z}G$, for a finite group $G$. Swan proved (see [CR1, 32.11]) that every finitely generated projective $\mathbb{Z}G$-module is locally free; so $Cl(\mathbb{Z}G) \cong \tilde{K}_0(\mathbb{Z}G)$. But group rings and rings of algebraic integers are special in this respect; $Cl$ and $\tilde{K}_0$ differ for $\mathbb{Z}$-orders in general. Between these two, it is $Cl$ that inherits the properties of $K_0$:

$$Cl(\Lambda_1 \oplus \Lambda_2) \cong Cl(\Lambda_1) \oplus Cl(\Lambda_2)$$

by [RU2], and

$$Cl(M_n(R)) \cong Cl(R)$$

by [Re, 36.6].

If $f : A_1 \to A_2$ is a $\mathbb{Q}$-algebra homomorphism carrying a $\mathbb{Z}$-order $\Lambda_1$ into a $\mathbb{Z}$-order $\Lambda_2$, the map $K_0(f) : K_0(\Lambda_1) \to K_0(\Lambda_2)$ induces a group homomorphism

$$Cl(f) : Cl(\Lambda_1) \to Cl(\Lambda_2)$$

making $Cl$ a functor. If $i : \Lambda \to \Lambda'$ is the inclusion of $\Lambda$ into a maximal $\mathbb{Z}$-order $\Lambda'$ of $A$ containing $\Lambda$, the map $Cl(i)$ is surjective; its kernel $D(\Lambda)$ is known as the *kernel group* of $\Lambda$, and up to isomorphism $D(\Lambda)$ is independent of the choice $\Lambda'$ (see [J]). So we have a (not necessarily split) short exact sequence

$$0 \to D(\Lambda) \to Cl(\Lambda) \xrightarrow{Cl(i)} Cl(\Lambda') \to 0. \tag{5}$$

So for a finite group $G$ and associated $\Lambda := \mathbb{Z}G$, understanding the group $\tilde{K}_0(\mathbb{Z}G) \cong Cl(\Lambda)$ boils down to understanding the groups $Cl(\Lambda')$ and $D(\Lambda)$ and the way these fit together.

Let us now specialize to the case of dihedral groups. For $G$ the dihedral group $D_n$, the isomorphism of $\mathbb{Q}$-algebras

$$\mathbb{Q}D_n \cong \bigoplus_{d \,|\, n, d>2} M_2(\mathbb{Q}(\zeta_d + \zeta_d^{-1})) \oplus \mathbb{Q}^{2\epsilon}$$

carries $\mathbb{Z}D_n$ into the maximal order

$$\Lambda' \cong \bigoplus_{d \,|\, n, d>2} M_2(\mathbb{Z}[\zeta_d + \zeta_d^{-1}]) \oplus \mathbb{Z}^{2\epsilon}.$$

Since $Cl(\mathbb{Z}) = 0$, we obtain

$$Cl(\Lambda') \cong \bigoplus_{d \,|\, n, d>2} Cl(\mathbb{Z}[\zeta_d + \zeta_d^{-1}]).$$

These summands, and the ideal class groups $Cl(\mathbb{Z}[\zeta_d])$, have been studied since the 19th century work of Kummer and Dedekind. They remain difficult to compute, and their orders

$$h_d = h\big(\mathbb{Q}(\zeta_d)\big) = |Cl(\mathbb{Z}[\zeta_d])|$$
$$h_d^+ = h\big(\mathbb{Q}(\zeta_d + \zeta_d^{-1})\big) = |Cl(\mathbb{Z}[\zeta_d + \zeta_d^{-1}])|$$

are still topics of active research.

A prime $p$ is *regular* if $p$ does not divide $h_p$, and *semiregular* if $p$ does not divide $h_p^+$. A conjecture originally discussed by Kummer, but currently known as Vandiver's Conjecture, is that all primes are semiregular. This has been verified for all primes less than 125,000 (see [Wag]). The smallest irregular prime is 37. Note that the order of $Cl(\Lambda')$ in the case $G = D_n$ is $\prod_d h_d^+$ for $d|n$, $d > 2$. If $d|n$, then $h_d^+|h_n^+$ (see [Le]); so $Cl(\Lambda') = 0$ if and only if $h_n^+ = 1$. Computer calculations show that $h_n^+ = 1$ for all $n \leq 71$ (see [Li]).

The kernel group $D(\mathbb{Z}D_n)$ vanishes for $n$ a prime [GRU], and for $n$ a power of a regular prime [FKW], [K]. Each surjective group homomorphism $G \to H$ induces a surjective homomorphism $D(\mathbb{Z}G) \to D(\mathbb{Z}H)$ by [RU2]. So if $d|n$, $D(\mathbb{Z}D_n)$ maps onto $D(\mathbb{Z}D_d)$. According to [EM2, Theorem 5.2], $D(\mathbb{Z}D_{p^2}) \cong (\mathbb{Z}_p)^d$ for all semiregular primes $p$, where $d > 1$ when $p$ is irregular. So $D(\mathbb{Z}D_n) \neq 0$ when $n$ is divisible by the square of an irregular, semiregular prime. Of course we can remove the "semiregular" condition if Vandiver's Conjecture is true.

Also in [EM2], $D(\mathbb{Z}D_n)$ is shown to have even order if (a) $n$ is divisible by three different odd primes, (b) $n$ is divisible by 4 and two different odd primes, or (c) $n$ is divisible by two different primes in $1+4\mathbb{Z}$ (in a parallel result by [Le], $h_n^+$ is even if $n$ is divisible by three distinct primes in $1+4\mathbb{Z}$). Another result in [EM2] is the proof that $D(\mathbb{Z}D_n) = 0$ for all $n < 60$.

Next let us consider the situation for groups of the form $G = D_n \times \mathbb{Z}_2$. Letting $\Lambda = \mathbb{Z}[D_n \times \mathbb{Z}_2]$, we again exploit the short exact sequence (5). Let $c \in \mathbb{Z}_2$ denote the non-trivial element in the cyclic group of order 2. Since $\mathbb{Q}[D_n \times \mathbb{Z}_2] \cong \mathbb{Q}D_n \oplus \mathbb{Q}D_n$ by an isomorphism $(c \to (1, -1))$ taking $\mathbb{Z}[D_n \times \mathbb{Z}_2]$ into $\mathbb{Z}D_n \oplus \mathbb{Z}D_n$, the $Cl(\Lambda')$ vanishes for $G = D_n \times \mathbb{Z}_2$ if and only if it vanishes for the corresponding $D_n$.

In contrast to $Cl(\Lambda')$, the computation of the kernel group $D(\Lambda)$ is much more involved. First of all, let us consider the case where $n$ is a power of 2. By [OT, Example 6.9], we know that $D(\mathbb{Z}[D_{2^r} \times \mathbb{Z}_2]) \cong \mathbb{Z}_{2^r}$. In fact, by [T], this group is the "Swan subgroup" generated by the class of the ideal $I$ in $\mathbb{Z}[D_{2^r} \times \mathbb{Z}_2]$ generated as an ideal by 5 and the sum of the elements in $D_{2^r} \times \mathbb{Z}_2$. For an alternative generator of this group, consider the cartesian square

$$\begin{tikzcd}
\mathbb{Z}[D_n \times \mathbb{Z}_2] \arrow[r] \arrow[d] & \mathbb{Z}[D_n] \arrow[d] \\
\mathbb{Z}[D_n] \arrow[r] & \mathbb{F}_2[D_n]
\end{tikzcd}$$

(with the left map $c \mapsto 1$, top map $c \mapsto -1$ and remaining maps reduction mod 2). Now by [RU2], the corresponding $K$-theory Mayer Vietoris sequence restricts to an exact sequence:

$$K_1(\mathbb{Z}D_n) \to K_1(\mathbb{F}_2[D_n]) \xrightarrow{\partial} D(\mathbb{Z}[D_n \times \mathbb{Z}_2]) \to (D(\mathbb{Z}[D_n]))^2 \to 0.$$

Now in the special case where $n = 2^r$, the term $(D(\mathbb{Z}[D_n]))^2$ vanishes. Through computations of $(\mathbb{F}_2 D_{2^r})^*$ one can show $D(\mathbb{Z}[D_{2^r} \times \mathbb{Z}_2])$ is also generated by $\partial(1 + b + ab)$. Finally, since $D(\mathbb{Z}[D_n \times \mathbb{Z}_2])$ maps onto $D(\mathbb{Z}[D_{2^r} \times \mathbb{Z}_2])$ if $2^r | n$, we see that $D(\mathbb{Z}[D_n \times \mathbb{Z}_2]) \neq 0$ for $n$ even.

Now for $n$ odd, we know that $D_n \times \mathbb{Z}_2 \cong D_{2n}$. If $p$ is an odd prime, $D(\mathbb{Z}[D_{2p}])$ is the cokernel of the map $R^* \to (R/2R)^*$, where $R = \mathbb{Z}[\zeta_d + \zeta_d^{-1}]$ (see [CR2, 50.14]). Generally , if $n$ is odd, the first map in the above sequence factors as the reduced norm, followed by reduction mod 2:

$$\bigoplus_{d|n, d>2} \mathbb{Z}[\zeta_d + \zeta_d^{-1}]^* \oplus \mathbb{Z}[b]^* \to \bigoplus_{d|n, d>2} \left( \frac{\mathbb{Z}[\zeta_d + \zeta_d^{-1}]}{2} \right)^* \oplus \mathbb{F}_2[b]^*$$

followed by an isomorphism to $K_1(\mathbb{F}_2 D_n)$. So for odd $n$, if the kernel of the surjective map

$$D(\mathbb{Z}[D_{2n}]) \to D(\mathbb{Z}[D_n])^2$$

is to be zero, then $R^* \to (R/2R)^*$ must be surjective for all $R = \mathbb{Z}[\zeta_d + \zeta_d^{-1}]$ with $d|n$, $d > 2$.

The simplest conclusion we can draw about $\tilde{K}_0(\mathbb{Z}D_n)$ is that it vanishes for $n < 60$. For a regular prime $p$, $\tilde{K}_0(\mathbb{Z}D_{p^r})$ vanishes whenever $h^+_{p^r} = 1$ (which may be true for all $r$ but is only known to be so for $\phi(p^r) \leq 66$). And $\tilde{K}_0(\mathbb{Z}D_n)$ has even order if $n$ is divisible by too many different primes $p$. Computer calculations are now accessible for $\tilde{K}_0(\mathbb{Z}G)$ for groups $G$ of modest size [BB].

3.4. **The Whitehead group $Wh(G)$.** From [Ba1] and [Wa] we know

$$K_1(\mathbb{Z}G) \cong \pm G^{ab} \oplus SK_1(\mathbb{Z}G) \oplus \mathbb{Z}^{r-q}$$

where $SK_1(\mathbb{Z}G)$ is finite and $r$ and $q$ are the numbers $r_\mathbb{R}$, $r_\mathbb{Q}$ of simple components of $\mathbb{R}G$, $\mathbb{Q}G$ respectively. From Berman's Theorem, $r_\mathbb{R}$ is the number of conjugacy classes of unordered pairs $\{x, x^{-1}\}$ with $x \in G$, and $r_\mathbb{Q}$ is the number of conjugacy classes of cyclic subgroups of $G$. Furthermore

$$Wh(G) = K_1(\mathbb{Z}G)/\{\pm G^{ab}\} = SK_1(\mathbb{Z}G) \oplus \mathbb{Z}^{r-q}.$$

For $G$ a dihedral group $D_n$ with $\epsilon$ conjugacy classes of reflections ($\epsilon$ =1 or 2 according to whether $n$ is odd or even), we computed in Section 3.1 that $q = \delta(n) + \epsilon$ where $\delta(n)$ is the number of divisors of $n$. Counting conjugacy classes of pairs $\{x, x^{-1}\}$ with $x \in D_n$, we find $r = (n + 3\epsilon)/2$. So $K_1(\mathbb{Z}D_n)$ and $Wh(D_n)$ have rank $(n + \epsilon)/2 - \delta(n)$. Now $(D_n)^{ab} \cong (\mathbb{Z}_2)^\epsilon$, and by [Ma1], $SK_1(\mathbb{Z}D_n) = 1$. So

$$Wh(D_n) \cong \mathbb{Z}^{(n+\epsilon)/2-\delta(n)},$$

$$K_1(\mathbb{Z}D_n) \cong \mathbb{Z}_2^{\epsilon+1} \oplus \mathbb{Z}^{(n+\epsilon)/2-\delta(n)}.$$

For $G = D_n \times \mathbb{Z}_2$, $F[D_n \times \mathbb{Z}_2] \cong (FD_n)^2$ for any coefficient field $F$ with $2 \neq 0$; so $r$, $q$ are doubled. Also $(D_n \times \mathbb{Z}_2)^{ab} \cong (D_n)^{ab} \times \mathbb{Z}_2$; and by [Ma2], $SK_1(\mathbb{Z}[D_n \times \mathbb{Z}_2]) = 1$. So

$$Wh(D_n \times \mathbb{Z}_2) \cong \mathbb{Z}^{n+\epsilon-2\delta(n)},$$

$$K_1(\mathbb{Z}[D_n \times \mathbb{Z}_2]) \cong \mathbb{Z}_2^{\epsilon+2} \oplus \mathbb{Z}^{n+\epsilon-2\delta(n)}.$$

This completes the computation of the lower algebraic $K$-theory of the cell stabilizers for the $\Gamma_P$-action on $\mathbb{H}^3$.

## 4. Homology of $E_{\mathcal{FIN}}\Gamma_P$

In order to simplify notation, we will omit the coefficients $\mathbb{K}\mathbb{Z}^{-\infty}$ in the equivariant homology theory, and will use $\Gamma$ to denote the Coxeter group $\Gamma_P$ associated to a finite volume geodesic polyhedron $P \subset \mathbb{H}^3$. Our goal in this section is to explain how to compute the term $H_n^\Gamma(E_{\mathcal{FIN}}\Gamma)$. First recall that the $\Gamma$ action on $\mathbb{H}^3$ provides a model for $E_{\mathcal{FIN}}$, with fundamental domain given by the original polyhedron $P$. If the polyhedron $P$ is non-compact, we can obtain a cocompact model for $E_{\mathcal{FIN}}\Gamma$ by equivariantly removing a suitable collection of horoballs from $\mathbb{H}^3$. A fundamental domain for this action is a copy of the polyhedron $P$ with each ideal vertex truncated. According to whether $P$ is compact or not, we will use $X$ to denote either $\mathbb{H}^3$, or $\mathbb{H}^3$ with the suitable horoballs removed. We will denote by $\hat{P}$ the quotient space $X/\Gamma$, a copy of $P$ with all ideal vertices truncated.

We observe that for this model, with respect to the obvious $\Gamma$-CW-structure, we have a very explicit description of cells in $X/\Gamma = \hat{P}$, as well as the corresponding stabilizers. The cells in $\hat{P}$ are of two distinct types. The first type of cells are cells from the original $P$; we call these type I cells. Namely:

- there is one 3-cell (the interior of $\hat{P}$) with trivial stabilizer,

- the 2-cells corresponding to faces of $P$, and they all have stabilizers isomorphic to $\mathbb{Z}_2$,
- the 1-cells corresponding to elements in $E(P)$, and their stabilizers will be finite dihedral groups, given by the special subgroup corresponding to the two faces intersecting in the given edge,
- the 0-cells corresponding to elements in $V(P)$, and their stabilizers will be 2-dimensional spherical Coxeter groups, given by the special subgroup corresponding to the three faces containing the given vertex.

In addition to these, we have cells arising from truncating ideal vertices in $P$, which we call type II cells. They are as follows:

- each truncated ideal vertex from $P$ gives rise to a 2-cell in $\hat{P}$, with trivial stabilizer,
- each face in $P$ incident to an ideal vertex gives rise to a 1-cell in $\hat{P}$ with stabilizer $\mathbb{Z}_2$,
- each edge in $P$ incident to an ideal vertex gives rise to a 0-cell in $\hat{P}$ with stabilizer a dihedral group (isomorphic to the stabilizer of the edge that is being truncated). From the fact that the ideal vertex stabilizers are 2-dimensional Euclidean reflection groups, the stabilizer can only be isomorphic to one of the groups $D_2, D_3, D_4$, or $D_6$.

Now to compute the homology group $H_n^{\Gamma}(X)$, we recall that Quinn has established [Qu, App. 2] the existence of an Atiyah-Hirzebruch type spectral sequence which converges to this homology group, with $E^2$-terms given by:

$$E^2_{p,q} = H_p(\hat{P}\,; \{Wh_q(\Gamma_\sigma)\}) \Longrightarrow H^{\Gamma}_{p+q}(X).$$

The complex that gives the homology of $\hat{P}$ with local coefficients $\{Wh_q(\Gamma_\sigma)\}$ has the form

$$\cdots \to \bigoplus_{\sigma^{p+1}} Wh_q(\Gamma_{\sigma^{p+1}}) \to \bigoplus_{\sigma^p} Wh_q(\Gamma_{\sigma^p}) \to \bigoplus_{\sigma^{p-1}} Wh_q(\Gamma_{\sigma^{p-1}}) \cdots \to \bigoplus_{\sigma^0} Wh_q(\Gamma_{\sigma^0}),$$

where $\sigma^p$ denotes the cells in dimension $p$, and the sum is over all $p$-dimensional cells in $\hat{P}$. The $p^{th}$ homology group of this complex will give us the entries for the $E^2_{p,q}$-term of the spectral sequence. Let us recall that

$$Wh_q(F) = \begin{cases} Wh(F), & q = 1 \\ \tilde{K}_0(\mathbb{Z}F), & q = 0 \\ K_q(\mathbb{Z}F), & q \leq -1. \end{cases}$$

Note that, from the description of the stabilizers given above, we know that there is only one 3-cell, with trivial stabilizer, and that all the 2-cells have stabilizers that are either trivial or isomorphic to $\mathbb{Z}_2$. But it is well known that the lower algebraic $K$-theory of both the trivial group and $\mathbb{Z}_2$ vanishes. In particular, for the groups of interest to us, we have that $E^2_{p,q} = 0$ except possibly for $p = 0, 1$. It is also a well-known result of Carter [C] that for a finite group $G$, $K_n(\mathbb{Z}G) = 0$ for $n < -1$. This tells us that the only possible non-zero values for $E^2_{p,q}$ occur when $p = 0, 1$ and $-1 \leq q \leq 1$, and are given by the homology of:

$$0 \to \bigoplus_{e \in E(P)} Wh_q(\Gamma_e) \to \bigoplus_{v \in V(P)} Wh_q(\Gamma_v) \to 0 \tag{6}$$

So in order to finish our computation of the $E^2$-terms, we merely need to find the various $Wh_q(\Gamma_e)$ and $Wh_q(\Gamma_v)$, and to analyze the morphism appearing above.

Recall that the edge stabilizers are given by dihedral groups $D_k$ (1-cells of type I), or are isomorphic to $\mathbb{Z}_2$ (1-cells of type II). Note that we have already largely computed the lower algebraic $K$-theory of dihedral groups (see Section 3). Concerning the vertex stabilizers, we note that these will be spherical triangle groups. The classification of these groups is well known: up to isomorphism, they are either the generic $D_k \times \mathbb{Z}_2$ ($k \geq 2$), or one of the three exceptional cases $S_4$, $S_4 \times \mathbb{Z}_2$, and $A_5 \times \mathbb{Z}_2$. We observe that, for the three exceptional cases, the lower algebraic $K$-theory has already been computed: we refer the reader to [LO2] for $S_4$, to [Or, Section 5] for $S_4 \times \mathbb{Z}/2$, and to [LO2, Section 5.4] and Section 3.2 for the group $A_5 \times \mathbb{Z}_2$. On the other hand, for the generic case, we have already given explicit computations for the lower algebraic $K$-theory (see Section 3).

4.1. **Analysis of the chain complex.** Now that we know the groups appearing in the chain complex (6), let us proceed to explain how one can compute the $E^2$-terms for the Quinn spectral sequence for $E_{\mathcal{FIN}}\,\Gamma$.

Recall that the only edges with potentially non-trivial $K$-groups are the edges of type I, with stabilizers $\Gamma_e$ isomorphic to dihedral groups. Each vertex in $\hat{P}$ has three incident edges. Vertices of type I have stabilizers $\Gamma_v$ which are spherical triangle groups and the inclusions $\Gamma_e \hookrightarrow \Gamma_v$ always corresponds to the inclusion of a special subgroup $\Gamma_e$ into the finite Coxeter group $\Gamma_v$. In contrast, vertices of type II have stabilizers $\Gamma_v$ which are dihedral; two incident edges are of type II with stabilizer isomorphic to $\mathbb{Z}_2$. The third incident edge is of type I, with stabilizer $G_e$ one of the dihedral groups $D_2, D_3, D_4$ or $D_6$, and with the inclusion $G_e \hookrightarrow G_v$ an isomorphism.

We now proceed to a case by case analysis based on the order of the edge stabilizers arising in the truncated polyhedron $\hat{P}$.

**Case 1: $\mathbf{n \geq 7}$.** If we have an edge $e \in E(\hat{P})$ with stabilizer $D_n$, $n \geq 7$, then both vertices $v, w$ appearing as endpoints of $e$ must be of type I, with stabilizer isomorphic to $D_n \times \mathbb{Z}_2$. Indeed, such $D_n$ do not appear as subgroups of any other spherical triangle group, nor do they appear as stabilizers of type II vertices. In this case, we observe that all the remaining edges incident to either $v$ or $w$ have to have stabilizers isomorphic to $D_2$, which we know has vanishing lower algebraic $K$-theory. This implies that for such an edge $e \in E(\hat{P})$, we can split off the portion of the chain complex (6) corresponding to $e$:

$$0 \to Wh_q(D_n) \to 2 \cdot Wh_q(D_n \times \mathbb{Z}_2) \to 0.$$

Furthermore, since $D_n \hookrightarrow D_n \times \mathbb{Z}_2$ is a retract, we see that the map above is injective, hence the homology will be concentrated in dimension zero, and will contribute a summand $2 \cdot Wh_q(D_n \times \mathbb{Z}_2)/Wh_q(D_n)$ to the corresponding $E^2_{0,q}$.

**Case 2: $\mathbf{n = 6}$.** If we have an edge $e \in E(\hat{P})$ with stabilizer $D_6$, the situation is a bit more complicated. The endpoints $v, w$ of the edge $e$ are either of type I (with vertex stabilizer $D_6 \times \mathbb{Z}_2$) or of type II (with vertex stabilizer $D_6$). In both cases, the remaining edges incident to the vertices $v, w$ have stabilizers isomorphic to $\mathbb{Z}_2$ or $D_2$, which we know have vanishing lower algebraic $K$-theory. So again, for each such edge $e \in E(\hat{P})$, we can split off the portion of the chain complex (6) corresponding to $e$:

$$0 \to Wh_q(D_6) \to Wh_q(\Gamma_v) \oplus Wh_q(\Gamma_w) \to 0.$$

We now consider each of the cases $q = 1, 0, -1$.

For $q = 1$, we have that $Wh(D_6)$ and $Wh(D_6 \times \mathbb{Z}_2)$ both vanish, so that the sequence above degenerates to the identically zero sequence. In particular, the edges with stabilizer $D_6$ do not contribute to $E^2_{1,1}$ or $E^2_{0,1}$.

For $q = 0$, we recall that $Wh_0(D_6) = 0$, while $Wh_0(D_6 \times \mathbb{Z}_2) \cong (\mathbb{Z}_2)^2$ (see [LO2, Section 5.1]. Hence each edge with stabilizer $D_6$ makes no contribution to $E^2_{1,0}$, while the contribution to $E^2_{0,0}$ is either 0, $(\mathbb{Z}_2)^2$, or $(\mathbb{Z}_2)^4$ according to whether none, one, or both of its vertices have stabilizer $D_6 \times \mathbb{Z}_2$.

Finally, for $q = -1$, we have that $Wh_{-1}(D_6) \cong \mathbb{Z}$ and $Wh_{-1}(D_6 \times \mathbb{Z}_2) \cong \mathbb{Z}^3$. Since the natural inclusion $D_6 \hookrightarrow D_6 \times \mathbb{Z}_2$ is a retract, the corresponding induced map on $Wh_{-1}$ is a split injection. This implies that edges with stabilizer $D_6$ do not contribute to the $E^2_{1,-1}$. At the level of $E^2_{0,-1}$, we find that an edge with stabilizer $D_6$ contributes either a $\mathbb{Z}$, $\mathbb{Z}^3$, or $\mathbb{Z}^5$, according to whether none, one, or both of its vertices have stabilizer $D_6 \times \mathbb{Z}_2$.

*Remark:* Let $r$ denote the number of vertices in $P$ with stabilizer $D_6 \times \mathbb{Z}_2$, and $E_6$ denotes the number of edges with stabilizer $D_6$. Then the overall non-trivial contribution from all the edges with stabilizer $D_6$ can be summarized as follows:

- a contribution of $\mathbb{Z}_2^{2r}$ to the $E^2_{0,0}$, and
- a contribution of $\mathbb{Z}^{E_6+2r}$ to the $E^2_{0,-1}$.

**Case 3: $\mathbf{n = 5}$.** If we have an edge $e \in E(\hat{P})$ with stabilizer $D_5$, then the two endpoints $v, w$ of the edge must be of type I. However, we still have two possibilities for the stabilizers of the two endpoints $v, w$. Indeed, the dihedral group $D_5$ appears as a special subgroup in two different spherical triangle groups: $D_5 \times \mathbb{Z}_2$, as well as in $[3, 5] \cong A_5 \times \mathbb{Z}_2$. Note that for the vertices with stabilizer $D_5 \times \mathbb{Z}_2$, the remaining incident edges will have stabilizers $D_2$, which we know has vanishing lower algebraic $K$-theory. On the other hand, vertices with stabilizer $A_5 \times \mathbb{Z}_2$ will have two additional incident edges, one with stabilizer $D_3$, and one with stabilizer $D_2$. But again, we know that these groups have vanishing lower algebraic $K$-theory. Hence we see that in all cases, we can split off the portion of the chain complex (6) corresponding to $e$:

$$0 \to Wh_q(D_5) \to Wh_q(\Gamma_v) \oplus Wh_q(\Gamma_w) \to 0.$$

Now recall that $Wh_q(D_5)$ vanishes, except for $q = 1$, where $Wh_1(D_5) \cong \mathbb{Z}$. For the group $D_5 \times \mathbb{Z}_2$, the non-vanishing lower algebraic $K$-groups consist of $Wh_1(D_5 \times \mathbb{Z}_2) \cong \mathbb{Z}^2$, and $Wh_{-1}(D_5 \times \mathbb{Z}_2) \cong \mathbb{Z}$. Finally, for the group $A_5 \times \mathbb{Z}_2$, all three lower $K$-groups are non-trivial, with $Wh_1(A_5 \times \mathbb{Z}_2) \cong \mathbb{Z}^2$, $Wh_0(A_5 \times \mathbb{Z}_2) \cong \mathbb{Z}_2$, and $Wh_{-1}(A_5 \times \mathbb{Z}_2) \cong \mathbb{Z}^2$.

Now for $q = 1$, the chain complex gives:

$$0 \to \mathbb{Z} \to \mathbb{Z}^2 \oplus \mathbb{Z}^2 \to 0$$

where the first $\mathbb{Z}$ comes from $Wh_1(D_5)$, and each $\mathbb{Z}^2$ comes from either a copy of $Wh_1(D_5 \times \mathbb{Z}_2)$ or a copy of $Wh_1(A_5 \times \mathbb{Z}_2)$. Note that since $D_5 \hookrightarrow D_5 \times \mathbb{Z}_2$ is a retract, the induced mapping of $\mathbb{Z} \to \mathbb{Z}^2$ on Whitehead groups is split injective. Furthermore, the authors have shown in [LO2, Section 7.3] that the map $\mathbb{Z} \to \mathbb{Z}^2$

on Whitehead groups induced by the inclusion $D_5 \hookrightarrow A_5 \times \mathbb{Z}_2$ is likewise split injective. Combining these two observations, we see that *regardless of the vertex stabilizers*, each edge with stabilizer $D_5$ will contribute a $\mathbb{Z}^3$ to the $E^2_{0,1}$, and will make no contribution to $E^2_{1,1}$.

Next we consider the case $q = 0$. The chain complex degenerates to:

$$0 \to Wh_0(\Gamma_v) \oplus Wh_0(\Gamma_w) \to 0.$$

This tells us that each edge with stabilizer $D_5$ makes no contribution to $E^2_{1,0}$. As for the contribution to $E^2_{0,0}$, each such edge contributes either a $0, \mathbb{Z}_2$, or $(\mathbb{Z}_2)^2$, according to whether none, one, or both of its vertices have stabilizer $A_5 \times \mathbb{Z}_2$.

Finally, we look at the case $q = -1$. Again, the chain complex degenerates to:

$$0 \to Wh_{-1}(\Gamma_v) \oplus Wh_{-1}(\Gamma_w) \to 0$$

giving us that edges with stabilizer $D_5$ make no contribution to $E^2_{1,-1}$. For the contribution to $E^2_{0,-1}$, we see that each such edge contributes either a $\mathbb{Z}^2, \mathbb{Z}^3$, or $\mathbb{Z}^4$, according to whether none, one, or both of its vertices have stabilizer $A_5 \times \mathbb{Z}_2$.

*Remark:* Let $s$ denote the number of vertices in $P$ with stabilizer $A_5 \times \mathbb{Z}_2$, and $E_5$ denote the number of edges with stabilizer $D_5$. Then the overall non-trivial contribution from all the edges with stabilizer $D_5$ can be summarized as follows:

- a contribution of $\mathbb{Z}^{3E_5}$ to the $E^2_{0,1}$,
- a contribution of $\mathbb{Z}_2^s$ to the $E^2_{0,0}$, and
- a contribution of $\mathbb{Z}^{2E_5+s}$ to the $E^2_{0,-1}$.

**Case 4: n = 4.** If we have an edge $e \in E(\hat{P})$ with stabilizer $D_4$, then we have three possibilities for the stabilizers of the two endpoints $v, w$. On the one hand, the vertex could be of type II, with stabilizer isomorphic to $D_4$. Among spherical triangle groups, $D_4$ appears as a special subgroup in only two different groups : $D_4 \times \mathbb{Z}_2$, and $[3, 4] \cong S_4 \times \mathbb{Z}_2$. So alternatively, we could have one or both endpoint vertices of type I, with stabilizer $D_4 \times \mathbb{Z}_2$ or $S_4 \times \mathbb{Z}_2$.

Now in all three cases, we see that the remaining incident edges to the vertices have stabilizers isomorphic to either $D_2$ or $D_3$, which have vanishing lower algebraic $K$-theory, so we can again split off the portion of the chain complex (6) corresponding to $e \in E(P)$. Observing that the group $D_4$ has no lower algebraic $K$-theory, the portion of the chain complex further degenerates into:

$$0 \to Wh_q(\Gamma_v) \oplus Wh_q(\Gamma_w) \to 0,$$

and hence there will be no contribution to $E^2_{1,1}$, $E^2_{1,0}$, and $E^2_{1,-1}$. Further observe that all three of the groups $D_4$, $D_4 \times \mathbb{Z}_2$ and $S_4 \times \mathbb{Z}_2$ have vanishing $Wh_1$. So no matter what the incident vertex groups are, we see that there is also no contribution to $E^2_{0,1}$ from the edges with stabilizer $D_4$.

Next let us consider what happens with $Wh_0$. Both $D_4 \times \mathbb{Z}_2$ and $S_4 \times \mathbb{Z}_2$ have $Wh_0$ isomorphic to $\mathbb{Z}_4$, while $D_4$ has vanishing $Wh_0$. In particular, we see that each edge with stabilizer $D_4$ will contribute 0, $\mathbb{Z}_4$, or $(\mathbb{Z}_4)^2$ to $E^2_{0,0}$, according to whether the edge joins two, one, or no ideal vertices.

The situation for $Wh_{-1}$ is likewise more complicated, as we have $Wh_{-1}(D_4 \times \mathbb{Z}_2) = 0$, while $Wh_{-1}(S_4 \times \mathbb{Z}_2) \cong \mathbb{Z}$. Hence the edge with stabilizer $D_4$ will contribute $0, \mathbb{Z}$, or $\mathbb{Z}^2$ to $E^2_{0,-1}$ according to whether it has none, one, or two of its vertices with stabilizer $S_4 \times \mathbb{Z}_2$.

*Remark:* Let $t$ denote the number of vertices in $P$ with stabilizer $S_4 \times \mathbb{Z}_2$, $u$ denote the number of ideal vertices with stabilizer $[4,4] = P4m$, and $E_4$ denote the number of edges with stabilizer $D_4$. Then the overall non-trivial contribution from all the edges with stabilizer $D_4$ can be summarized as follows:

- a contribution of $\mathbb{Z}_4^{2E_4-2u}$ to the $E^2_{0,0}$, and
- a contribution of $\mathbb{Z}^t$ to the $E^2_{0,-1}$.

**Case 5: $\mathbf{n \leq 3}$.** For edges $e \in E(\hat{P})$ with stabilizer $D_3$ or $D_2$, the contribution to the $E^2$-terms in the Quinn spectral sequence is concentrated on those vertices with stabilizer $D_3 \times \mathbb{Z}_2$ or $D_2 \times \mathbb{Z}_2$. Indeed, we have on the one hand that the lower algebraic $K$-theory of the edge groups $D_3$ and $D_2$ vanish, so the contribution to the $E^2$-terms will come solely from the corresponding vertex groups. The contribution from the vertices having an incident edge with stabilizer $D_n$, $n \geq 4$, has already been accounted for (in the appropriate case above). So we are left with dealing with vertices, all of whose incident edges are either $D_3$ or $D_2$. The only such vertices have stabilizer $S_4$, $D_3 \times \mathbb{Z}_2$, or $D_2 \times \mathbb{Z}_2$. Amongst these, the only non-vanishing $K$-theory appears for $D_3 \times \mathbb{Z}_2 \cong D_6$ (with $K_{-1}$ isomorphic to $\mathbb{Z}$) and $D_2 \times \mathbb{Z}_2$ (with $\tilde{K}_0$ isomorphic to $\mathbb{Z}_2$).

*Remark:* Let $v$ denote the number of vertices in $P$ with stabilizer $D_3 \times \mathbb{Z}_2$, and $w$ denote the number of vertices with stabilizer $D_2 \times \mathbb{Z}_2$ Then the overall non-trivial contribution from all the edges with stabilizer $D_3$ and $D_2$ can be summarized as follows:

- a contribution of $\mathbb{Z}_2^w$ to the $E^2_{0,0}$, and
- a contribution of $\mathbb{Z}^v$ to the $E^2_{0,-1}$.

4.2. **Collapsing of the spectral sequence and applications.** Now collecting the information from the previous few sections, we immediately see that the $E^2$-terms in the Quinn spectral sequence all vanish, with the possible exception of $E^2_{0,1}$, $E^2_{0,0}$, and $E^2_{0,-1}$ (within the range $q \leq 1$).

In particular, the spectral sequence always collapses at the $E^2$-stage, and yields the desired homology group. Furthermore, from the analysis in the previous section, we obtain (see the Remarks after Cases 2, 3, 4, and 5) the following explicit formulas:

$$H_1^{\Gamma}(X; \mathbb{KZ}^{-\infty}) \cong \mathbb{Z}^{3E_5} \oplus Q_1$$

$$H_0^{\Gamma}(X; \mathbb{KZ}^{-\infty}) \cong (\mathbb{Z}_2)^{2r+s+w} \oplus (\mathbb{Z}_4)^{2E_4-2u} \oplus Q_0$$

$$H_{-1}^{\Gamma}(X; \mathbb{KZ}^{-\infty}) \cong \mathbb{Z}^{2r+s+t+v+2E_5+E_6} \oplus Q_{-1}$$

where in the expression above we have that:

- $r$ is the number of special subgroups isomorphic to $D_6 \times \mathbb{Z}_2$,
- $s$ is the number of special subgroups isomorphic to $A_5 \times \mathbb{Z}_2$,
- $t$ is the number of special subgroups isomorphic to $S_4 \times \mathbb{Z}_2$,
- $u$ is the number of ideal vertices in $P$ with stabilizer $P4m$,
- $v$ is the number of special subgroups isomorphic to $D_3 \times \mathbb{Z}_2$,
- $w$ is the number of special subgroups isomorphic to $D_2 \times \mathbb{Z}_2$,
- $E_4, E_5$, and $E_6$ are the number of edges in $P$ with stabilizer $D_4, D_5$, and $D_6$ respectively.

and the terms $Q_q$ are given by:

$$Q_q \cong \bigoplus_{e \in E_l(P)} \frac{2 \cdot Wh_q(\Gamma_e \times \mathbb{Z}_2)}{Wh_q(\Gamma_e)}$$

where $E_l(P)$ denotes the subset of edges of $P$ having "large" stabilizer, i.e. satisfying $\Gamma_e = D_n$ with $n \geq 7$.

Now let us discuss some applications of these spectral sequence computations. Recall that the Farrell-Jones isomorphism conjecture holds for the groups $\Gamma_P$, and hence the lower algebraic $K$-theory $Wh_*(\Gamma_P)$ of $\Gamma_P$ can be identified with $H_*^\Gamma(E_{\mathcal{VC}}\,\Gamma_P; \mathbb{KZ}^{-\infty})$. Furthermore, the term $H_*^\Gamma(E_{\mathcal{FIN}}\,\Gamma_P; \mathbb{KZ}^{-\infty})$ computed above is a direct summand inside $H_*^\Gamma(E_{\mathcal{VC}}\,\Gamma_P; \mathbb{KZ}^{-\infty})$, and hence a direct summand inside $Wh_*(\Gamma_P)$ (see equation (2) in Section 2).

For $* = 0, 1$, the remaining terms in equation (2) are known to be *purely torsion*, and in particular, vanish when we tensor with $\mathbb{Q}$. Specializing to $* = 1$, and keeping the notation from above, we immediately obtain that

$$Wh(\Gamma_P) \otimes \mathbb{Q} = \mathbb{Q}^{3E_5} \oplus (Q_1 \otimes \mathbb{Q}).$$

Along with the computations in Section 3.4, this allows us to explicitly determine the rationalized Whitehead group:

**Theorem 5.** *Let $\Gamma_P$ be a hyperbolic reflection group with associated finite volume geodesic polyhedron $P \subset \mathbb{H}^3$. Then the rationalized Whitehead group has rank:*

$$rk\big(Wh(\Gamma_P) \otimes \mathbb{Q}\big) = \frac{3}{2} \sum_n E_n \big[n + \epsilon(n) - 2\delta(n)\big] \tag{7}$$

*where $E_n$ is the number of edges in $P$ with stabilizer $D_n$, $\epsilon(n)$ equals 1 or 2 according to whether $n$ is odd or even, and $\delta(n)$ is the number of divisors of $n$.*

*Proof.* By the discussion above, we need to analyze the term $Q_1 \otimes \mathbb{Q}$. For a given edge with stabilizer $D_n$ ($n \geq 7$), we see a contribution of $2rk(Wh(D_n \times \mathbb{Z}_2)) - rk(Wh(D_n))$ to the overall rank of $Q_1 \otimes \mathbb{Q}$. Appealing to the ranks of $Wh$ computed in section 3.4, we see that such an edge contributes

$$2\big(n + \epsilon - 2\delta(n)\big) - \big((n+\epsilon)/2 - \delta(n)\big) = (3/2) \cdot \big(n + \epsilon - 2\delta(n)\big)$$

to the rank of $Q_1 \otimes \mathbb{Q}$. Summing over all edges with stabilizer $D_n$, $n \geq 7$, and adding in the contribution from the edges with stabilizer $D_5$, we obtain that:

$$rk(Wh(\Gamma_P) \otimes \mathbb{Q}) = 3E_5 + \frac{3}{2} \sum_{n \geq 7} E_n \big[n + \epsilon(n) - 2\delta(n)\big].$$

To conclude, we merely observe that for $n = 2, 3, 4, 6$, the expression $n + \epsilon(n) - 2\delta(n)$ equals zero, while for $n = 5$, we have $5 + \epsilon(5) - 2\delta(5) = 5 + 1 - 2(2) = 2$. So we see that the expression computed above for $rk(Wh(\Gamma_P) \otimes \mathbb{Q})$ is in fact equal to the expression appearing in equation (7), concluding the proof. □

Next, let us consider the case $* = -1$. In this case, it is known that the remaining terms in the splitting given in equation (2) *all vanish*. In particular, this gives us isomorphisms

$$K_{-1}(\mathbb{Z}\,\Gamma_P) \cong H_{-1}^\Gamma(X; \mathbb{KZ}^{-\infty}) \cong \mathbb{Z}^{2r+s+t+2E_5+E_6} \oplus Q_{-1}.$$

Furthermore, we have explicit computations (see Theorem 1) for the various $K$-groups appearing in the description of $Q_{-1}$. Substituting in those calculations, we immediately obtain:

**Theorem 6.** *Let $\Gamma_P$ be a hyperbolic reflection group with associated finite volume geodesic polyhedron $P \subset \mathbb{H}^3$. Then the group $K_{-1}(\mathbb{Z}\Gamma_P)$ is torsion-free, with rank given by the expression:*

$$2r + s + t + v + 2E_5 + E_6 + \sum_{n\geq 7} E_n\big(1 - 3\delta(n) + 3\tau(n) + 2\sigma_2(n)\big)$$

*where $r, s, t, v$ are the number of vertex stabilizers isomorphic to $D_6 \times \mathbb{Z}_2$, $A_5 \times \mathbb{Z}_2$, $S_4 \times \mathbb{Z}_2$, and $D_3 \times \mathbb{Z}_2$ respectively, the $E_k$ are the number of edges in $P$ with stabilizer $D_k$, and the number theoretic quantities $\delta(n), \tau(n), \sigma_2(n)$ are as defined in Section 3.1.*

Finally, let us make a few comments on the case $* = 0$. In this situation, we cannot deduce any similar nice formulas for the $\tilde{K}_0(\mathbb{Z}\Gamma_P)$, the difficulties being twofold. On the one hand, the computation of $H_0^{\Gamma}(E_{\mathcal{VC}}\,\Gamma_P; \mathbb{KZ}^{-\infty})$ involves knowing the reduced $\tilde{K}_0$ for dihedral groups and products of dihedral groups with $\mathbb{Z}_2$. As we say in Section 3.3, these computations are closely related to some difficult questions in algebraic number theory, and always yield torsion groups. On the other hand, the remaining terms in the expression for $\tilde{K}_0(\mathbb{Z}\Gamma_P)$ (see expression (2) in Section 2) can sometimes be non-zero (see Section 5), and are likewise (infinitely generated) torsion groups. Since it is known that these remaining terms are also purely torsion, we can only conclude that the group $\tilde{K}_0(\mathbb{Z}\Gamma_P)$ is a torsion group (which is already known to follow from the Farrell-Jones isomorphism conjecture for $\Gamma_P$).

## 5. Cokernels of relative assembly maps for $V \in \mathcal{V}$

In this section, we focus on understanding the second term appearing in the splitting formula given in equation (2). We recall that this term is of the form:

$$\bigoplus_{V\in\mathcal{V}} H_n^V(E_{\mathcal{FIN}}(V) \to *) \tag{8}$$

where $\mathcal{V}$ consists of one representative from each conjugacy class of the infinite groups that arise as a stabilizers of single geodesics in $\mathbb{H}^3$, and $H_n^V(E_{\mathcal{FIN}}(V) \to *)$ is the cokernel of the maps on homology $H_n^V(E_{\mathcal{FIN}}(V); \mathbb{KZ}^{-\infty}) \to H_n^V(*; \mathbb{KZ}^{-\infty})$, which we call the *relative assembly map.*

Let $\gamma$ be a geodesic giving rise to a summand in expression (8). Since the stabilizer of $\gamma$ is assumed to be infinite, we conclude that $\mathrm{Stab}(\gamma)$ acts cocompactly on $\gamma$, and hence the projection $\pi(\gamma)$ of $\gamma$ to the fundamental domain $P$ is compact. There are three possibilities for the projection $\pi(\gamma)$:

- either it intersects the interior of $P$,
- it lies entirely in the 2-skeleton of $P$, and intersects the interior of a face,
- it lies entirely in the 1-skeleton of $P$.

The argument given by Lafont-Ortiz in [LO1, Prop. 3.5, 3.6] applies verbatim to show that in the first two cases, the stabilizer of the geodesic $\gamma$ has to be isomorphic to one of the groups $\mathbb{Z}$, $D_\infty$, $\mathbb{Z} \times \mathbb{Z}_2$, or $D_\infty \times \mathbb{Z}_2$. Now for all four of these infinite groups, it is well known that the cokernel of the relative assembly map is trivial (see [Ba2], [Wd] for the first two, and [Pe] for the last two). In particular, these groups will make *no contribution* to the expression (8).

So let us consider geodesics of the third type. First of all, note that two such geodesics $\gamma_1, \gamma_2$ will have $Stab(\gamma_1)$ conjugate to $Stab(\gamma_2)$ if and only if $\pi(\gamma_1) = \pi(\gamma_2)$ (as subsets of $P$). In particular, we see that among the groups in $\mathcal{V}$, there are *at most finitely many* groups of this type. Indeed, since there are exactly $|E(P)| < \infty$ edges in the 1-skeleton of $P$, we can have at most $|E(P)|$ such subgroups (up to conjugacy) inside $\Gamma_P$. In particular, the infinite direct sum in expression (8) really collapses down to a finite direct sum.

We now focus on identifying (1) the actual number of such subgroups, and (2) the corresponding cokernels for the relative assembly map. In order to complete this process, we first observe the following: for any such group, we can consider the action on the corresponding geodesic $\gamma$, obtaining a splitting

$$0 \to \mathrm{Fix}_\Gamma(\gamma) \to \mathrm{Stab}_\Gamma(\gamma) \to \mathrm{Isom}_{\Gamma,\gamma}(\mathbb{R}) \to 0$$

where $\mathrm{Fix}_\Gamma(\gamma)$ is the subgroup of $\Gamma$ fixing $\gamma$ pointwise, while $\mathrm{Isom}_{\Gamma,\gamma}(\mathbb{R})$ is the induced action of the stabilizer $\mathrm{Stab}_\Gamma(\gamma)$ on $\mathbb{R}$ (identified with the geodesic $\gamma$). Note that since $\mathrm{Stab}_\Gamma(\gamma)$ is known to act discretely on $\mathbb{H}^3$, and cocompactly on $\gamma$, we immediately obtain that $\mathrm{Isom}_{\Gamma,\gamma}(\mathbb{R})$ is a discrete, cocompact subgroup of $\mathrm{Isom}(\mathbb{R})$, i.e. has to be isomorphic to $\mathbb{Z}$ or $D_\infty$. On the other hand, the term $\mathrm{Fix}_\Gamma(\gamma)$ corresponds to the subset fixing $\gamma$ pointwise, and taking a point in $\gamma$ that projects to the *interior* of an edge $e$, we immediately see that this group must be isomorphic to a dihedral group $D_n$ (coinciding with the stabilizer of the edge $e$). As in the previous section, let us proceed with a case by case analysis, according to the order of the group $\mathrm{Fix}_\Gamma(\gamma)$.

**Case 1: $\mathbf{n \geq 6}$.** If we have a geodesic $\gamma$ with infinite stabilizer, satisfying $\mathrm{Fix}_\Gamma(\gamma) \cong D_n$ with $n \geq 6$, then we observe that $\pi(\gamma) \subset P$ coincides with a single edge in the 1-skeleton of $P$ (see [LO2, Section 4]), with stabilizer $D_n$. Furthermore, both vertex endpoints of the edge must be non-ideal, with vertex stabilizer isomorphic to $D_n \times \mathbb{Z}_2$.

It is easy to see that the vertex stabilizers actually leave the geodesic $\gamma$ invariant. So applying Bass-Serre theory, we see that to each edge of $P$ with internal angle $\pi/n$ with $n \geq 6$, one has an element in $\mathcal{V}$ isomorphic to $(D_n \times \mathbb{Z}_2) *_{D_n} (D_n \times \mathbb{Z}_2) \cong D_n \times D_\infty$. We note that for $V$ of the form $D_n \times D_\infty$, the cokernel of the relative assembly map satisfies:

$$H_*^V(E_{\mathcal{FIN}}(V) \to *) \cong NK_*(\mathbb{Z}D_n)$$

where $NK_*(\mathbb{Z}D_n)$ is the Bass Nil-group associated to the dihedral group $D_n$ (see [D], [DKR], [DQR]). In particular we see that *each geodesic extending an edge with stabilizer $D_n$, $n \geq 6$, joining non-ideal vertices (in the case $n = 6$), will contribute a single copy of the Bass Nil-group for $D_n$.*

**Case 2: $\mathbf{n = 5}$.** If we have a geodesic $\gamma$ with the property that $\mathrm{Fix}_\Gamma(\gamma) \cong D_5$, then we observe that, once again, the projection $\pi(\gamma)$ of the geodesic into the polyhedron $P$ will consist of a single edge with stabilizer $D_5$ (see [LO2, Section 4]). Note that the endpoints of this edge must have stabilizer either $D_5 \times \mathbb{Z}_2$, or $A_5 \times \mathbb{Z}_2$.

Again, this allows us to use Bass-Serre theory to identify the stabilizer of $\gamma$. It will be an amalgamation of two finite groups over the common (index two) subgroup

$D_5$. Furthermore, the vertex groups correspond precisely to the subgroups of the vertex stabilizer that *also leaves $\gamma$ invariant*. It is easy to check that, regardless of whether the vertex stabilizer is $D_5 \times \mathbb{Z}_2$ or $A_5 \times \mathbb{Z}_2$, this subgroup has to be isomorphic to $D_{10} \cong D_5 \times \mathbb{Z}_2$. We conclude that each geodesic with $\mathrm{Fix}_\Gamma(\gamma) \cong D_5$ must have stabilizer isomorphic to $D_{10} *_{D_5} D_{10} \cong D_5 \times D_\infty$. The cokernel of this relative assembly map is known to be isomorphic to the Bass Nil-group $NK_*(\mathbb{Z}D_5)$ ([LO3], see also [DKR]), which are known to vanish for $* \leq 1$. We conclude that *geodesics extending edges with stabilizer $D_5$ make no contribution to the lower algebraic $K$-theory.*

**Case 3: $\mathbf{n = 4}$.** If we have a geodesic $\gamma$ with the property that $\mathrm{Fix}_\Gamma(\gamma) \cong D_4$, then we observe that, once again, the projection $\pi(\gamma)$ of the geodesic into the polyhedron $P$ will consist of a single edge with stabilizer $D_4$. In this case, the two endpoints of this edge must have stabilizer either isomorphic to $D_4 \times \mathbb{Z}_2$ or to $S_4 \times \mathbb{Z}_2$. In both cases, one can see that the subgroup of the vertex stabilizers that also leaves the geodesic invariant are isomorphic to $D_4 \times \mathbb{Z}_2$. Hence, we obtain that the stabilizer of $\gamma$ is an amalgamation $(D_4 \times \mathbb{Z}_2) *_{D_4} (D_4 \times \mathbb{Z}_2) \cong D_4 \times D_\infty$. We have discussed this cokernel in [LO2, Section 6.4]: it can be identified with the Bass Nil groups $NK_*(\mathbb{Z}D_4)$ (see also [D], [DKR], [DQR]). In particular, we see that *each geodesic extending an edge with stabilizer $D_4$, and with infinite stabilizer, will contribute a single copy of the Bass Nil group for $D_4$.*

*Remark:* We note that these Nil-groups have been partially computed by Weibel [We], who showed that $NK_0(\mathbb{Z}D_4)$ is isomorphic to the direct sum of a countably infinite free $\mathbb{Z}/2$-module with a countably infinite free $\mathbb{Z}/4$-module. He also showed that $NK_1(\mathbb{Z}D_4)$ is a countably infinite torsion group of exponent 2 or 4.

**Case 4: $\mathbf{n = 3}$.** Geodesics $\gamma$ with the property that $\mathrm{Fix}_\Gamma(\gamma) \cong D_3$ are somewhat more difficult to track. The reason for this is that an edge in $P$ with stabilizer $D_3$ can have four possible stabilizers for the endpoints. Indeed, the spherical triangle groups containing $D_3$ as a special subgroup include $D_3 \times \mathbb{Z}_2$, $S_4$, $S_4 \times \mathbb{Z}_2$, and $A_5 \times \mathbb{Z}_2$.

Now if the projection $\pi(\gamma)$ of the geodesic is a union of edges forming an interval, then we can use Bass-Serre theory to write out the stabilizer of the geodesic. From the tessellations associated to the four possible vertex stabilizers, we can readily see that the geodesic is reflected whenever the endpoint has stabilizer $D_3 \times \mathbb{Z}_2$, $S_4 \times \mathbb{Z}_2$, or $A_5 \times \mathbb{Z}_2$. In all three cases, one sees that the subgroup of the vertex stabilizer that leaves the $\gamma$ invariant is in fact isomorphic to $D_3 \times \mathbb{Z}_2$. We conclude that in this case, the stabilizer of $\gamma$ is isomorphic to $(D_3 \times \mathbb{Z}_2) *_{D_3} (D_3 \times \mathbb{Z}_2) \cong D_3 \times D_\infty$. But the authors have shown that the cokernel of the relative assembly map for this group is isomorphic to the Bass Nil-group $NK_*(\mathbb{Z}D_3)$ (see [LO1, Section 5], as well as [D], [DKR], [DQR]).

Alternatively, the projection $\pi(\gamma)$ of the geodesic could be a union of edges forming a closed loop in the 1-skeleton $P^{(1)}$ of $P$. In this case, we see that the stabilizer of $\gamma$ fits into a short exact sequence:

$$0 \to D_3 \to \mathrm{Stab}_\Gamma(\gamma) \to \mathbb{Z} \to 0$$

and hence can be written as a semidirect product $D_3 \rtimes_\alpha \mathbb{Z}$. In this case, the cokernel of the relative assembly map will be a Farrell Nil-group $NK_i(\mathbb{Z}D_3, \alpha)$.

Summarizing this discussion, we see that each orbit of a periodic geodesic in $\mathbb{H}^3$ which is pointwise fixed by a $D_3$ contributes a single copy of a Farrell Nil-group $NK_*(\mathbb{Z}D_3;\alpha)$ (for a suitable automorphism $\alpha \in \mathrm{Aut}(D_3)$). Finally, we remark that for $* = 0, 1$, the Farrell Nil-groups $NK_*(\mathbb{Z}D_3;\alpha)$ are known to vanish, irrespective of the automorphism $\alpha$. Hence we obtain that *the geodesics extending edges with stabilizer $D_3$ make no contribution to the lower algebraic $K$-theory.*

**Case 5: $\mathbf{n = 2}$.** Geodesics $\gamma$ with $\mathrm{Fix}_\Gamma(\gamma) \cong D_2$ are the most difficult ones to handle. The primary difficulty is that every spherical triangle group contains $D_2$ as a special subgroup. Now assume we have such a geodesic $\gamma$, and consider its projection into the 1-skeleton $P^{(1)}$ of $P$. The projection is either:

- a union of edges forming a closed loop inside $P^{(1)}$, or
- a union of edges forming a path inside $P^{(1)}$.

If the projection is a path, Bass-Serre theory applies, and the stabilizer of the geodesic $\gamma$ will have to be isomorphic to one of the groups $D_4 *_{D_2} D_4$, $D_4 *_{D_2} (D_2 \times \mathbb{Z}_2)$, or $(D_2 \times \mathbb{Z}_2) *_{D_2} (D_2 \times \mathbb{Z}_2) \cong D_2 \times D_\infty$ (depending on the nature of the endpoints of the path). In this situation, the authors have established (see [LO1, Section 4] and [LO2, Sections 6.2, 6.3]) that for all three of these groups, the cokernels of the relative assembly map are isomorphic to the Bass Nil-group $NK_*(\mathbb{Z}D_2)$ corresponding to the canonical index two subgroup isomorphic to $D_2 \times \mathbb{Z}$. These Bass Nil-groups are known to be isomorphic to $\bigoplus_\infty \mathbb{Z}_2$, a countable direct sum of $\mathbb{Z}_2$, in dimensions $* = 0$ and $* = 1$.

Alternatively, if the projection is a closed loop, then from the short exact sequence:

$$0 \to D_2 \to \mathrm{Stab}_\Gamma(\gamma) \to \mathbb{Z} \to 0$$

we have that the stabilizer is of the form $D_2 \rtimes_\alpha \mathbb{Z}$, $\alpha \in \mathrm{Aut}(D_2)$. We now claim that the geometry of the situation forces $\alpha = Id$, i.e. the stabilizer is in fact a direct product $D_2 \times \mathbb{Z}$. In order to see this, we first observe that $\mathrm{Aut}(D_2) = \mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong S_3$, given by an arbitrary permutation of the three non-zero elements in $D_2$. Let us try to rule out the various automorphisms in $\mathrm{Aut}(D_2)$.

First, let us denote by $g, h$ the reflections in the hyperplanes $P_1, P_2$ extending the two faces of the polyhedron incident to one of the edges in the closed loop. Now the fixed subgroup of $\gamma$ can be identified with the subgroup $D_2$ consisting of $\{1, g, h, gh\}$, and the elements $g, h$ are the canonical generators of the special subgroup $D_2$. We also have that $\alpha$ permutes the subset $\{g, h, gh\}$. But observe that $g, h$ are reflections, whereas their product $gh$ is a rotation by $\pi$ around the geodesic $\gamma$. Since rotations are never conjugate to reflections, this implies that $gh$ must be fixed by the permutation $\alpha$. So the only possibility that is left is where $\alpha$ interchanges $g$ and $h$.

In order to rule out this last possibility, we can look at the element $\tau$ in the stabilizer of $\gamma$ that acts via a minimal translation along $\gamma$. Note that $\tau$ either maps each $P_i$ to itself, or interchanges $P_1$ and $P_2$. So to rule out the case where $\alpha$ interchanges the two reflections $g$ and $h$, it is sufficient to identify the element $\tau$, and verify that it leaves invariant each of the hyperplanes $P_i$.

We now focus on explicitly describing the element $\tau$ in the group $\Gamma$. Given a pair of consecutive edges in this loop, we have that the corresponding common vertex of intersection must have stabilizer $\Gamma_v$ of the form $D_k \times \mathbb{Z}_2$, with $k$ an *odd* integer. The two incoming edges with stabilizer $D_2$ correspond to the special subgroups $\Gamma_{e_i}$,

$G_{e_j}$ of $D_k \times \mathbb{Z}_2$ of the form $\langle g\rangle \times \mathbb{Z}_2$, where $g$ is one of the two canonical reflections generating $D_k$. In this situation, there is a unique element $\nu(v, e_i)$ of the group $\Gamma_v$ whose action takes the geodesic extending the edge $e_i$ to the geodesic extending the edge $e_j$. Of course, we have the obvious relation $\nu(v, e_j) = \nu(v, e_i)^{-1}$. In concrete terms, the element $\nu(v, e_i)$ can be described as follows: it is simply the longest word in the group $\Gamma_V$ (with respect to the Coxeter generating set). Note that the element $\nu(v, e_i)$ is always a rotation inside $\mathrm{Isom}(\mathbb{H}^3)$. Geometrically, this rotation of $\mathbb{H}^3$ fixes the vertex $v$, and at the level of the spherical tessellation of the unit tangent sphere at $v$, takes the spherical triangle corresponding to the polyhedron $P$ to the spherical triangle which is directly opposite.

Now assume that the loop of interest is given cyclicly by the sequence of edges and vertices $\{e_1, v_1, e_2, v_2, \ldots, e_n, v_n\}$. We can consider each of the elements $\nu(v_i, e_i)$, and observe that the product

$$\nu(v_1, e_1) \cdot \nu(v_2, e_2) \cdot \ldots \cdot \nu(v_n, e_n) \in \Gamma$$

clearly stabilizes the geodesic $\gamma$ extending the edge $e_1$. Furthermore, this element acts via a minimal translation along $\gamma$, and hence can be taken as an explicit description of the desired element $\tau$. Note that Deodhar [De] considered similar elements in the general setting of Coxeter groups (see also Davis' book [Da, Section 4.10]). We now are left with verifying that this element $\tau$ leaves each of the two hyperplanes $P_1, P_2$ invariant.

To check this last statement, we first consider how the element $\nu(v, e_j)$ acts on the hyperplanes whose intersection defines the vertex $v$. We note that there are three such hyperplanes $P_1, P_2, P_3$, labelled so that $P_1 \cap P_2 = e_j$ and $P_2 \cap P_3 = e_i$. In particular, we have that the hyperplanes $P_1$ and $P_3$ intersect at an angle $\pi/k$, with $k$ *odd*. Now from the explicit formula for $\nu(v, e_j)$, it is immediate that it leaves $P_2$ invariant, and interchanges $P_1$ and $P_3$. In other words, the element $\nu(v, e_j)$ interchanges the two hyperplanes whose intersection is the edge with internal dihedral angle $\pi/k$. We would now like to use this to compute the effect of the element $\tau$ on the original pair of hyperplanes.

We note that the loop of interest is a simple loop in the 1-skeleton $P^{(1)}$ of the polyhedron $P$, hence can be thought of as a simple closed curve on $\partial P \cong S^2$. In particular, this loop separates $\partial P$ into precisely two connected components $U_1, U_2$, and without loss of generality, we have that the faces $F_1, F_2$ whose intersection forms the edge $e_1$ satisfy $F_i \subset U_i$. Considering vertex $v_1$, let us think of the action of $\nu(v_1, e_1)$ on the two hyperplanes $P_1, P_2$. Since $v_1$ has degree three, we have an edge $f_1$ in $P^{(1)}$ incident to the loop, and this edge must have internal angle of the form $\pi/k$ ($k$ odd). Observe that $f_1$ is contained in one of the two components $U_1, U_2$. From the discussion in the previous paragraph, the effect of $\nu(v_1, e_1)$ is to interchange the hyperplanes extending the faces adjacent to $f_1$, and to leave invariant the hyperplane extending the face opposite $f_1$. But observe that the two faces incident to $f_1$ are contained in the same component $U_i$, while the opposite face to $f_1$ is contained in the other component. This forces the action of $\nu(v_1, e_1)$ to respect the components $U_1$, $U_2$. Similarly, we see that each of the elements $\nu(v_i, e_i)$ respect the individual components, which forces their product $\tau$ to similarly respect the components. Since $\tau$ maps the hyperplane $P_1$ extending $F_1$ to the hyperplane extending a face which:

- is incident to $e_1$, i.e. is either $F_1$ or $F_2$, and

- is in the same connected component $U_1$ as $F_1$

we conclude that $\tau$ leaves $P_1$ (and likewise $P_2$) invariant. This forces $\alpha = Id$, ensuring that the stabilizer of the corresponding geodesic must be isomorphic to the direct product $D_2 \times \mathbb{Z}$. For this group, the cokernel of the relative assembly map is the classic Bass Nil-group $NK_*(\mathbb{Z}D_2)$.

Finally, let us comment on the number of copies of this Bass Nil-group that will appear in our computation. This requires counting $\Gamma_P$ orbits of geodesics whose stabilizer is infinite, and is fixed by a subgroup isomorphic to $D_2$. But this is actually not too difficult. Indeed, such a geodesic has to project to the subset of the 1-skeleton of $P$ consisting of edges with internal dihedral angle $= \pi/2$. So given the polyhedron $P$, restrict to this subset of the 1-skeleton $P^{(1)}$, obtaining a graph $\mathcal{G}_2$. Since the 1-skeleton of $P$ has the property that every vertex has degree $\leq 4$, the subgraph $\mathcal{G}_2$ inherits this same property. Now disconnect this graph along all vertices of degree 3 or 4, resulting in a collection of intervals and loops. Finally, disconnect the graph at any vertex of degree 2, having the property that the *third incident edge in* $\mathcal{G}$ has internal dihedral angle which is *even*. Discard all intervals with the property that one of their endpoints came from a vertex of degree 4. Then there is a bijective correspondence between:

(1) connected components of the resulting graph $\hat{\mathcal{G}}_2$,
(2) $\Gamma_p$-orbits of geodesics $\gamma \subset \mathbb{H}^3$ with infinite stabilizer and $\mathrm{Fix}_\Gamma(\gamma) \cong D_2$.

By the discussion in the last couple of pages, we conclude that *geodesics extending the edges with stabilizer $D_2$ contribute a total of $|\pi_0(\hat{\mathcal{G}}_2)| \cdot NK_*(\mathbb{Z}D_2)$ to the lower algebraic $K$-theory.*

*Remark:* We note that for $i = 0, 1$ each of these Bass Nil-groups $NK_i(\mathbb{Z}D_2)$ is isomorphic to $\bigoplus_\infty \mathbb{Z}_2$, the direct sum of countably many copies of $\mathbb{Z}_2$ (see [LO1, Lemma 5.3, 5.4]).

## 6. Appendix: concrete examples.

To illustrate the methods discussed in this paper, we now proceed to work through the lower algebraic $K$-theory for some concrete examples. Let us start with a relatively simple class of examples. Consider the groups $\Lambda_n$, $n \geq 5$, given by the following presentation:

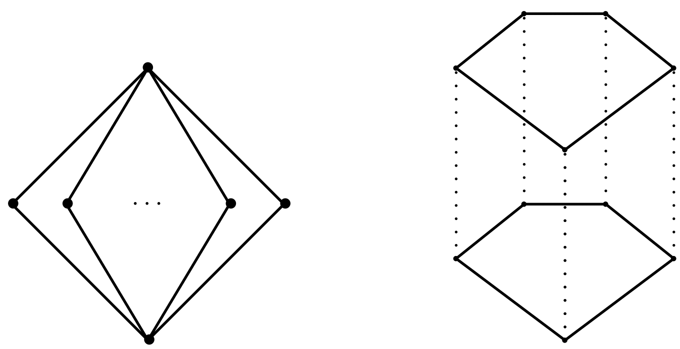
$$\Lambda_n := \left\langle y, z, x_i,\ \ 1 \leq i \leq n \ \middle|\ \begin{array}{c} y^2, z^2, \\ x_i^2, (x_i x_{i+1})^2, (x_i z)^3, (x_i y)^3, \ \ 1 \leq i \leq n \end{array} \right\rangle$$

The groups $\Lambda_n$ are Coxeter groups, and the presentation given above is in fact the Coxeter presentation of the group. The corresponding Coxeter diagram appears in Figure 1(a).

**Example 7.** *For the groups $\Lambda_n$ whose presentations are given above,*

*(1) the Whitehead group is given by*

$$Wh(\Lambda_n) \cong n \cdot NK_1(D_2) \cong \bigoplus_\infty \mathbb{Z}_2;$$



(a) (b)

Figure 1

*(2) the* $\tilde{K}_0$ *is given by*

$$\tilde{K}_0(\mathbb{Z}\Lambda_n) \cong n \cdot NK_0(D_2) \cong \bigoplus_{\infty} \mathbb{Z}_2;$$

*(3) the* $K_{-1}$ *always vanishes.*

*Proof.* The groups $\Lambda_n$ arise as hyperbolic reflection groups, with underlying polyhedron $P$ the product of an $n$-gon with an interval. An illustration of the polyhedron associated to the group $\Lambda_5$ is shown in Figure 1(b), where again, ordinary edges have dihedral angle $\pi/3$, while dotted edges have dihedral angle $\pi/2$. In general, the polyhedron associated to the group $\Lambda_n$ is combinatorially a product of the $n$-gon with an interval. This polyhedron has exactly two faces which are $n$-gons, and the dihedral angle along the edges of these two faces is $\pi/3$. All the remaining edges have dihedral angle $\pi/2$.

To begin with, we observe that for the associated polyhedron, every edge has stabilizer $D_2$ or $D_3$, giving us $E_k = 0$ for $k \geq 4$. Furthermore, for the associated polyhedron, every vertex has stabilizer $S_4$, implying that $r = s = t = v = 0$. Applying Theorem 6, we immediately obtain that $K_{-1}(\mathbb{Z}\Lambda_n) = 0$. Applying Theorem 5, we also obtain that $Wh(\Lambda_n) \otimes \mathbb{Q} = 0$. Note that the discussion in Section 4.2 actually establishes that

$$H_1^{\Lambda_n}(E_{\mathcal{FIN}}\Lambda_n; \mathbb{K}\mathbb{Z}^{-\infty}) = 0$$

So to complete the computation of $Wh(\Lambda_n)$, we need to identify the remaining terms in the splitting described in equation (2). From the discussion in our Section 5, we see that the next step is to understand geodesics in $\mathbb{H}^3$ whose projection under the $\Lambda_n$-action lies in the 1-skeleton of the polyhedron $P$. But it is easy to see that, up to the $\Gamma_P$-action, these give:

- two distinct geodesics with stabilizer $D_3 \times \mathbb{Z}$, which project to the boundary of the two $n$-gons appearing in $P$, and

- $n$ distinct geodesics whose fixed subgroup is $D_2$, each of which projects to a single edge lying between the two $n$-gons in the polyhedron $P$.

Now we know (Section 5, Case 4) that geodesics with stabilizer $D_3 \times \mathbb{Z}$ yield no contribution to the splitting in equation (2). On the other hand, each of the geodesics with fixed subgroup $D_2$ contributes (Section 5, Case 5) a copy of $NK_1(\mathbb{Z}D_2)$, which is isomorphic to a countable infinite direct sum of $\mathbb{Z}_2$.

Finally, let us consider the case of $\tilde{K}_0$. Since we have $r = s = u = w = E_4 = 0$, we have (see Section 4.2) that $H_0^{\Lambda_n}(E_{\mathcal{FIN}}\Lambda_n; \mathbb{KZ}^{-\infty}) = 0$. Now the discussion in the previous paragraph, combined with the splitting in equation (2), gives us that $\tilde{K}_0(\mathbb{Z}\Lambda_n) \cong n \cdot NK_0(\mathbb{Z}D_2)$. But it is known that these Bass Nil-groups are isomorphic to the countable direct sum of infinitely many copies of $\mathbb{Z}_2$, concluding our computation.

□

Next, let us consider a somewhat more complicated family of examples. For an integer $n \geq 2$, we consider the group $\Gamma_n$, defined by the following presentation:

$$\Gamma_n := \left\langle x_1, \ldots, x_6 \;\middle|\; \begin{array}{c} x_i^2, (x_1x_2)^n, (x_1x_5)^2, (x_1x_6)^2, (x_3x_4)^2, (x_2x_5)^2, (x_2x_6)^2 \\ (x_1x_4)^3, (x_2x_3)^3, (x_4x_5)^3, (x_4x_6)^3, (x_3x_5)^3, (x_3x_6)^3 \end{array} \right\rangle$$

Observe that the groups $\Gamma_n$ are Coxeter groups, and that the presentation given above is in fact the Coxeter presentation of the group. The corresponding Coxeter diagram appears in Figure 2(a).

**Example 8.** *For the groups $\Gamma_n$ whose presentations are given above,*

*(1) the rationalized Whitehead group is given by*

$$Wh(\Gamma_n) \otimes \mathbb{Q} \cong \mathbb{Q}^{(3/2)\cdot(n+\epsilon(n)-2\delta(n))}$$

*(2) the Whitehead group is given by*

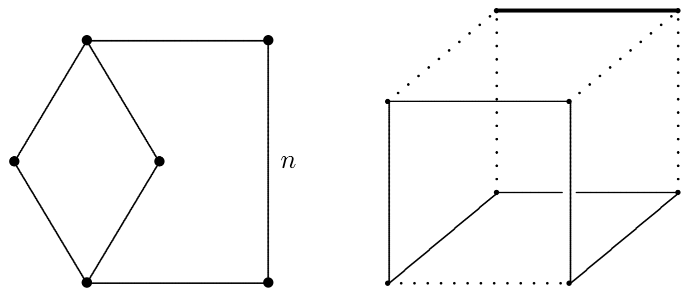
$$Wh(\Gamma_n) \cong \mathbb{Z}^{(3/2)\cdot(n+\epsilon(n)-2\delta(n))} \oplus (1+2\epsilon(n)) \cdot NK_1(\mathbb{Z}D_2) \oplus NK_1(\mathbb{Z}D_n)$$

*(3) the $\tilde{K}_0$ is given by*

$$\tilde{K}_0(\mathbb{Z}\,\Gamma_n) \cong \begin{cases} \frac{2\cdot\tilde{K}_0(\mathbb{Z}[D_n\times\mathbb{Z}_2])}{\tilde{K}_0(\mathbb{Z}[D_n])} \oplus (1+2\epsilon(n)) \cdot NK_0(\mathbb{Z}D_2) \oplus NK_0(\mathbb{Z}D_n) & n \geq 7 \\ \mathbb{Z}_2^4 \oplus 5 \cdot NK_0(\mathbb{Z}D_2) \oplus NK_0(\mathbb{Z}D_6) & n = 6 \\ \mathbb{Z}_4 \oplus 3 \cdot NK_0(\mathbb{Z}D_2) & n = 5 \\ \mathbb{Z}_4 \oplus 5 \cdot NK_0(\mathbb{Z}D_2) & n = 4 \\ 3 \cdot NK_0(\mathbb{Z}D_2) & n = 3 \\ \mathbb{Z}_2 \oplus 6 \cdot NK_0(\mathbb{Z}D_2) & n = 2 \end{cases}$$

*(4) the $K_{-1}$ is given by*

$$K_{-1}(\mathbb{Z}\,\Gamma_n) \cong \begin{cases} \mathbb{Z}^{1-3\delta(n)+3\tau(n)+2\sigma_2(n)} & n \geq 7 \\ \mathbb{Z}^5 & n = 6 \\ \mathbb{Z}^2 & n = 5 \\ 0 & n = 4 \\ \mathbb{Z}^2 & n = 3 \\ 0 & n = 2 \end{cases}$$




(a) (b)

FIGURE 2

*Proof.* To verify the results stated in this example, we first observe that the Coxeter groups $\Gamma_n$ arise as hyperbolic reflection groups, with underlying polyhedron $P$ a combinatorial cube. The geodesic polyhedron associated to $\Gamma_n$ is shown in the Figure 2(b). In the illustration, the bold edge has internal dihedral angle $\pi/n$, the ordinary edges have internal dihedral angle $\pi/3$, and the dotted edges have internal dihedral angle $\pi/2$.

To compute the rationalized Whitehead group, we just apply our Theorem 5. The polyhedron $P$ has five edges with stabilizer $D_2$, six edges with stabilizer $D_3$, and one edge with stabilizer $D_n$. Evaluating equation (7) gives us that the rank of $Wh(\Gamma_n) \otimes \mathbb{Q}$ is equal to $(3/2) \cdot (n + \epsilon(n) - 2\delta(n))$.

Now while Theorem 5 gives us a simple formula for the *rationalized* Whitehead group, it only requires a little bit more work to calculate the integral Whitehead group. In order to do this, we exploit the splitting given in equation (2) (see Section 2.4):

$$Wh(\Gamma_n) \cong H_1^{\Gamma_n}(E_{\mathcal{FIN}}(\Gamma_n); \mathbb{KZ}^{-\infty}) \oplus \bigoplus_{V \in \mathcal{V}} H_1^V(E_{\mathcal{FIN}}(V) \to *).$$

For the groups $\Gamma_n$, the first term in the splitting is computed in Section 4.2 (see the argument for Theorem 5), and is free abelian of rank $(3/2) \cdot (n + \epsilon(n) - 2\delta(n))$. As far as the second term in the splitting is concerned, we apply the procedure in Section 5. The edge with stabilizer $D_n$ contributes a single Bass Nil-group $NK_1(\mathbb{Z}D_n)$ to the splitting. The collection of edges with stabilizer $D_3$ form a closed cycle, which is the image of a single geodesic in $\mathbb{H}^3$. This gives rise to a single Farrell Nil-group $NK_1(\mathbb{Z}D_3, \alpha)$ (for a suitable automorphism $\alpha \in \mathrm{Aut}(D_3)$); but these groups are known to vanish. Finally, the edges with stabilizer $D_2$ correspond to either three or five geodesics in $\mathbb{H}^3$, according to whether $n$ is odd or even. Overall, these contribute $1 + 2\epsilon(n)$ copies of the Bass Nil-group $NK_1(\mathbb{Z}D_2)$ to $Wh(\Gamma_n)$. This completes our computation of $Wh(\Gamma_n)$.

Next, let us compute $K_{-1}(\mathbb{Z}\Gamma_n)$. We first observe that six of the eight vertices in $P$ have stabilizer $S_4$, while the remaining two vertices have stabilizer $D_n \times \mathbb{Z}_2$. This tells us that, in the notation of Theorem 6, $s = t = 0$. Now:

- if $n = 2$, then we additionally have that $E_k = 0$ for $k \geq 5$, and $r = v = 0$,
- if $n = 3$, then we have that $E_k = 0$ for $k \geq 5$, and $r = 0$, while $v = 2$,
- if $n = 4$, then we have that $E_k = 0$ for $k \geq 5$, and $r = v = 0$,
- if $n = 5$, then we have $E_k = 0$ for $k \geq 6$, $E_5 = 1$, and $r = v = 0$,
- if $n = 6$, then we have $v = E_5 = 0$, $E_6 = 1$, $E_k = 0$ for $k \geq 7$, and $r = 2$,
- if $n \geq 7$, then we have $r = v = E_5 = E_6 = 0$, and within the range $k \geq 7$, all the $E_k$ vanish except $E_n = 1$.

Applying Theorem 6 completes the computation of $K_{-1}(\mathbb{Z}\Gamma_n)$.

Lastly, let us compute $\tilde{K}_0(\mathbb{Z}\Gamma_n)$. We again make use of the splitting given in equation (2). The first term in the splitting, $H_0^{\Gamma_n}(E_{\mathcal{FIN}}(\Gamma_n); \mathbb{KZ}^{-\infty})$, is given by a particularly simple expression (see Section 4.2). Observe that, regardless of the value of $n$, we have that $s = u = 0$; so the expression for the first term in the splitting reduces to

$$H_0^{\Gamma_n}(E_{\mathcal{FIN}}(\Gamma_n); \mathbb{KZ}^{-\infty}) \cong (\mathbb{Z}_2)^{2r+w} \oplus (\mathbb{Z}_4)^{2E_4} \oplus Q_0$$

As such, we see that the first term vanishes, except in the following three cases:

- $n = 2$, where $r = E_4 = 0$, and $w = 1$, so the first term is $\mathbb{Z}_2$
- $n = 4$, where $r = w = 0$, $E_4 = 1$, and $Q_0 \cong 0$, so the first term is $\mathbb{Z}_4$,
- $n = 6$, where $r = 2$, $w = E_4 = 0$, and $Q_0 \cong 0$, and hence the first term is $\mathbb{Z}_2^4$,
- $n \geq 7$, where $r = w = E_4 = 0$, and hence the first term coincides with the group $Q_0 = 2\tilde{K}_0(\mathbb{Z}[D_n \times \mathbb{Z}_2])/\tilde{K}_0(\mathbb{Z}D_n)$.

For the second term in the splitting (2), we apply the methods from Section 5. The second term is determined by orbits of geodesics in $\mathbb{H}^3$ whose stabilizer is infinite, and which project to the 1-skeleton of the geodesic polyhedron associated to the group $\Gamma_n$. But these geodesics were determined earlier, when we discussed the computation of the Whitehead group. An identical analysis gives us that the second term is just $(1 + 2\epsilon(n)) \cdot NK_0(\mathbb{Z}D_2) \oplus NK_0(\mathbb{Z}D_n)$. Combining these two terms completes the computation of the $\tilde{K}_0$, and concludes the computations for this example. ☐

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