

Rational Equivariant K -Homology of Low Dimensional Groups

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ABSTRACT. We consider groups G which have a cocompact, 3-manifold model for the classifying space $\underline{E}G$. We provide an algorithm for computing the rationalized equivariant K -homology of $\underline{E}G$. Under the additional hypothesis that the quotient 3-orbifold $\underline{E}G/G$ is geometrizable, the rationalized K -homology groups coincide with the groups $K_*(C_{red}^*G) \otimes \mathbb{Q}$. We illustrate our algorithm on some concrete examples.

1. Introduction

We consider groups G which have a cocompact, 3-manifold model for the classifying space $\underline{E}G$. For such groups, we are interested in computing the equivariant K -homology of $\underline{E}G$. We develop an algorithm to compute the *rational* equivariant K -homology groups. If in addition we assume that the quotient 3-orbifold $\underline{E}G/G$ is geometrizable, then G satisfies the Baum-Connes conjecture, and the rational equivariant K -homology groups coincide with the groups $K_*(C_{red}^*G)$. These are the rationalized (topological) K -theory groups of the reduced C^* -algebra of G .

Some general recipes exist for computing the rational K -theory of an arbitrary group (see Lück and Oliver [LuO], as well as Lück [Lu1], [Lu2]). These general recipes pass via the Chern character. They typically involve identifying certain conjugacy classes of cyclic subgroups, their centralizers, and certain (group) homology computations. Similar formulas (with similar ingredients) appear in p -adic K -theory, after tensoring with \mathbb{Q}_p (see for instance Adem [Ad]).

In contrast, our methods rely instead on the low-dimensionality of the model for the classifying space $\underline{E}G$. Given a description of the model space $\underline{E}G$, our procedure is entirely algorithmic, and returns the ranks of the K -homology groups.

Let us briefly outline the contents of this paper. In Section 2, we provide some background material. Section 3 is devoted to explaining our algorithm, and the requisite proofs showing that the algorithm gives the desired K -groups. In Section 4 we implement our algorithm on several concrete classes of examples. Section 5 has some concluding remarks.

2. Background material

2.1. C^* -algebra. Given any discrete group G , one can form the associated reduced C^* -algebra. This Banach algebra is obtained by looking at the action $g \mapsto \lambda_g$ of G on the Hilbert space $l^2(G)$ of square summable complex-valued functions on G , given by the left regular representation:

$$\lambda_g \cdot f(h) = f(g^{-1}h) \quad g, h \in G, \quad f \in l^2(G).$$

The algebra $C_r^*(G)$ is defined to be the operator norm closure of the linear span of the operators λ_g inside the space $B(l^2(G))$ of bounded linear operators on $l^2(G)$. The Banach algebra $C_r^*(G)$ encodes various analytic properties of the group G .

2.2. Topological K -theory. For a C^* -algebra A , the corresponding (topological) K -theory groups can be defined in the following manner. The group $K_0(A)$ is defined to be the Grothendieck completion of the semi-group of finitely generated projective A -modules (with group operation given by direct sum). Since the algebra A comes equipped with a topology, one has an induced topology on the space $GL_n(A)$ of invertible $(n \times n)$ -matrices with entries in A , and as such one can consider the group $\pi_0(GL_n(A))$ of connected components of $GL_n(A)$ (note that this is indeed a group, not just a set). The group $K_1(A)$ is defined to be $\lim \pi_0(GL_n(A))$, where the limit is taken with respect to the sequence of natural inclusions of $GL_n(A) \hookrightarrow GL_{n+1}(A)$. The higher K -theory groups $K_q(A)$ are similarly defined to be $\lim \pi_{q-1}(GL_n(A))$, for $q \geq 2$. Alternatively, one can identify the functors $K_q(A)$ for all $q \in \mathbb{Z}$ via Bott 2-periodicity in q , i.e. $K_q(A) \cong K_{q+2}(A)$ for all q .

2.3. Baum-Connes conjecture. Let us now recall the statement of the Baum-Connes conjecture (see [BCH], [DL]). Given a discrete group G , there exists a specific generalized equivariant homology theory having the property that, if one evaluates it on a point $*$ with trivial G -action, the resulting homology groups satisfy $H_n^G(*) \cong K_n(C_r^*(G))$. Now for any G -CW-complex X , one has an obvious equivariant map $X \rightarrow *$. It follows from the basic properties of equivariant homology theories that there is an induced *assembly map*:

$$H_n^G(X) \rightarrow H_n^G(*) \cong K_n(C_r^*(G)).$$

Associated to a discrete group G , we have a classifying space for proper actions $\underline{E}G$. The G -CW-complex $\underline{E}G$ is well-defined up to G -equivariant homotopy equivalence, and is characterized by the following two properties:

- if $H \leq G$ is any infinite subgroup of G , then $\underline{E}G^H = \emptyset$, and
- if $H \leq G$ is any finite subgroup of G , then $\underline{E}G^H$ is contractible.

The Baum-Connes conjecture states that the assembly map

$$H_n^G(\underline{E}G) \rightarrow H_n^G(*) \cong K_n(C_r^*(G))$$

corresponding to $\underline{E}G$ is an isomorphism. For a thorough discussion of this topic, we refer the reader to the book by Mislin and Valette [MV] or the survey article by Lück and Reich [LuR].

2.4. 3-orbifold groups. We are studying groups G having a cocompact 3-manifold model for \underline{EG} . Let X denote this specific model for the classifying space, and for this section, we will further assume that the quotient 3-orbifold X/G is *geometrizable*.

The validity of the Baum-Connes conjecture for fundamental groups of orientable 3-manifolds has been established by Matthey, Oyono-Oyono, and Pitsch [MOP, Thm. 1.1] (see also [MV, Thm. 5.18] or [LuR, Thm. 5.2]). The same argument works in the context of geometrizable 3-orbifolds. We provide some details for the convenience of the reader.

LEMMA 1. *The Baum-Connes conjecture holds for the orbifold fundamental group of geometrizable 3-orbifolds.*

PROOF. In fact, the stronger *Baum-Connes property with coefficients* holds for this class of groups. This property states that a certain assembly map, associated to a G -action on a separable C^* -algebra A , is an isomorphism (and recovers the classical Baum-Connes conjecture when $A = \mathbb{C}$). The coefficients version has better inheritance properties, and in particular, is known to be inherited under graph of groups constructions (amalgamations and HNN-extensions), see Oyono-Oyono [O-O, Thm. 1.1].

The orbifold fundamental group of a geometrizable 3-orbifold can be expressed as an iterated graph of groups, with all initial vertex groups being orbifold fundamental groups of geometric 3-orbifolds. Geometric 3-orbifolds are cofinite volume quotients of one of the eight 3-dimensional geometries. Combined with Oyono-Oyono's result, the Lemma reduces to establishing the property for the orbifold fundamental group of finite volume geometric 3-orbifolds.

The fundamental work of Higson and Kasparov [HK] established the Baum-Connes property with coefficients for all groups satisfying the Haagerup property. We refer the reader to the monograph [CCJJV] for a detailed exposition of the Haagerup property. We will merely require the fact that groups acting with cofinite volume on all eight 3-dimensional geometries (\mathbb{E}^3 , S^3 , $S^2 \times \mathbb{E}^1$, \mathbb{H}^3 , $\mathbb{H}^2 \times \mathbb{E}^1$, $\widetilde{PSL}_2(\mathbb{R})$, Nil , and Sol) always have the Haagerup property, which will conclude the proof of the Lemma.

For the five geometries \mathbb{E}^3 , S^3 , $S^2 \times \mathbb{E}^1$, Nil , and Sol , any group acting on these will be amenable, and hence satisfy the Haagerup property. Lattices inside groups locally isomorphic to $SO(n, 1)$ are Haagerup (see [CCJJV, Thm. 4.0.1]), and hence groups acting on the two geometries \mathbb{H}^3 and $\widetilde{PSL}_2(\mathbb{R})$ are Haagerup. Finally, the Haagerup property is inherited by amenable extensions of Haagerup groups (see [CCJJV, Example 6.1.6]). This implies that groups acting on $\mathbb{H}^2 \times \mathbb{E}^1$ are Haagerup, for any such group is a finite extension of a group which splits as a product of \mathbb{Z} with a lattice in $SO(2, 1)$. This concludes the proof of the Lemma. \square

Remark: If one assumes that the G -action is smooth and orientation preserving, then Thurston's geometrization conjecture (now a theorem) predicts that X/G is a geometrizable 3-orbifold. The proof of the orbifold version of the conjecture was originally outlined by Thurston, and was independently established by Boileau, Leeb, and Porti [BLP] and Cooper, Hodgson, and Kerckhoff [CHK] (both loosely following Thurston's approach). The manifold version of the conjecture (i.e. trivial isotropy groups) is of course due to the recent work of Perelman.

Remark: If the quotient space X/G is not known to be geometrizable (for instance, if the G -action is not smooth, or does not preserve the orientation), then the argument in Lemma 1 does not apply. Nevertheless, our algorithm can still be used to compute the rational equivariant K -homology of $\underline{E}G$. It is however no longer clear that this coincides with $K(C_r^*(G)) \otimes \mathbb{Q}$.

2.5. Polyhedral CW-structures. Let us briefly comment on the G -CW-structure of X . As the quotient space X/G is a connected 3-orbifold, we can assume without loss of generality that the CW-structure contains a single orbit of 3-cell. Taking a representative 3-cell σ for the unique 3-cell orbit, we observe that the closure of σ must *contain* representatives of each lower dimensional orbit of cells. Indeed, if some lower dimensional cell had no orbit representatives contained in $\bar{\sigma}$, then there would be points in that lower dimensional cell with no neighborhood homeomorphic to \mathbb{R}^3 . Pulling back the 2-skeleton of the CW-structure via the attaching map of the 3-cell σ , we obtain (i) a decomposition of the 2-sphere into the pre-images of the individual cells, and (ii) an equivalence relation on the 2-sphere, identifying together points which have the same image under the attaching map. We note that the quotient space X/G can be reconstructed from this data. If in addition we know the isotropy subgroups of points, then X itself can be reconstructed from X/G . We will assume that we are given the G -action on X , in the form of a partition and equivalence relation on the 2-sphere as above, along with the isotropy data.

In some cases, one can find a G -CW-structure which is particularly simple: the 2-sphere coincides with the boundary of a polyhedron, the partition of the 2-sphere is into the faces of the polyhedron, and the equivalence relation linearly identifies together faces of the polyhedron. More precisely, we make the:

DEFINITION 2. A *polyhedral* CW-structure is a CW-structure where each cell is identified with the interior of a polyhedron $P_i \cong \mathbb{D}^k$, and the attaching map from the boundary $\partial\mathbb{D}^k \cong \partial P_i$ of a k -cell to the $(k-1)$ -skeleton, when restricted to each s -dimensional face of ∂P_i , is a combinatorial homeomorphism onto an s -cell in the $(k-1)$ -skeleton.

In the case where there is a polyhedral G -CW-structure on X with a single 3-cell orbit, then our algorithms are particularly easy to implement. All the concrete examples we will see in Section 4 come equipped with a polyhedral G -CW-structure.

Remark: It seems plausible that, if a G -CW-structure exists for a (topological) G -action on a 3-manifold X , then a polyhedral G -CW-complex structure should also exist. It also seems likely that, if a polyhedral G -CW-structure exists, then the G -action on the 3-manifold X should be smoothable.

For some concrete examples of polyhedral G -CW-structures, consider the case where X is either hyperbolic space \mathbb{H}^3 or Euclidean space \mathbb{R}^3 , and the G -action is via isometries. Then the desired G -equivariant polyhedral CW-complex structure can be obtained by picking a suitable point $p \in X$, and considering the Voronoi diagram with respect to the collection of points in the orbit $G \cdot p$. Another example, where X is the 3-dimensional Nil-geometry, is discussed in Section 4.2.

3. The algorithm

In this section, we describe the algorithm used to perform our computations. Throughout this section, let G be a group with a smooth action on a 3-manifold, providing a model for $\underline{E}G$. We will assume that $\underline{E}G$ supports a polyhedral G -CW-structure, and that P is a fundamental domain for the G -action on X , as described in Section 2.5. So P is the polyhedron corresponding to the single 3-cell orbit, and the orbit space $\underline{B}G$ is obtained from P by identifying various boundary faces together. We emphasize that the polyhedral G -CW-structure assumption serves only to facilitate the exposition: the algorithm works equally well with an arbitrary G -CW-structure.

3.1. Spectral sequence analysis. As explained in the previous section, the Baum-Connes conjecture provides an isomorphism:

$$H_n^G(\underline{E}G) \rightarrow H_n^G(*) \cong K_n(C_r^*(G)).$$

We are interested in computing the equivariant homology group arising on the left hand side of the assembly map. Since our group G is 3-dimensional, we will let X denote the 3-dimensional manifold model for $\underline{E}G$. To compute the equivariant homology of X , one can use an Atiyah-Hirzebruch spectral sequence. Specifically, there exists a spectral sequence (see [DL], or [Q, Section 8]), converging to the group $H_n^G(X)$, with E^2 -terms obtained by taking the homology of the following chain complex:

$$(1) \quad \cdots \rightarrow \bigoplus_{\sigma \in (X/G)^{(p+1)}} K_q(C_r^*(G_\sigma)) \rightarrow \bigoplus_{\sigma \in (X/G)^{(p)}} K_q(C_r^*(G_\sigma)) \rightarrow \cdots$$

In the above chain complex, $(X/G)^{(i)}$ consists of i -dimensional cells in the quotient X/G , or equivalently, G -orbits of i -dimensional cells in X . The groups G_σ denote the stabilizer of a cell in the orbit σ . Since our space X is 3-dimensional, we see that our chain complex can only have non-zero terms in the range $0 \leq p \leq 3$ (the morphisms in the chain complex will be described later, see Section 3.3). Moreover, since X is a model for $\underline{E}G$, all the cell stabilizers G_σ must be finite subgroups of G . For F a finite group, the groups $K_q(C_r^*(F))$ are easy to compute:

$$K_q(C_r^*(F)) = \begin{cases} 0 & \text{if } q \text{ is odd,} \\ \mathbb{Z}^{c(F)} & \text{if } q \text{ is even.} \end{cases}$$

Here, $c(F)$ denotes the number of conjugacy classes of elements in F . In fact, for q even, $K_q(C_r^*(F))$ can be identified with the complex representation ring of F . This immediately tells us that $E_{pq}^2 = 0$ for q odd. We will denote by \mathcal{C} the chain complex in equation (1) corresponding to the case where q is even. By the discussion above, we know that $H_p(\mathcal{C}) = 0$ except possibly in the range $0 \leq p \leq 3$. We summarize this discussion in the:

Fact 1: The only potentially non-vanishing terms on the E^2 -page (and hence any E^k -page, $k \geq 2$) occur when $0 \leq p \leq 3$ and q is even.

Next we note that the differentials on the E^k -page of the spectral sequence have bidegree $(-k, k + 1)$, i.e. are of the form $d_{p,q}^k : E_{p,q}^k \rightarrow E_{p-k,q+k-1}^k$. When $k = 2$, alternating rows on the E^2 -page are zero (see **Fact 1**), which implies that $E_{p,q}^3 = E_{p,q}^2$. When $k = 3$, the differentials $d_{p,q}^3$ shift horizontally by three units,

and up by two units. So the only potentially non-zero differentials on the E^3 -page are (up to vertical translation by the 2-periodicity in q) those of the form

$$d_{3,0}^3 : E_{3,0}^2 \cong E_{3,0}^3 \rightarrow E_{0,2}^3 \cong E_{0,0}^2.$$

Once we have $k \geq 4$, the differentials $d_{p,q}^k$ shift horizontally by $k \geq 4$ units. But **Fact 1** tells us that the only non-zero terms occur in the vertical strip $0 \leq p \leq 3$, which forces $E_{p,q}^4 \cong E_{p,q}^5 \cong \dots$ for all p, q . In other words, the spectral sequence collapses at the E^4 -stage. Since the E^2 -terms are given by the homology of \mathcal{C} , this establishes:

LEMMA 3. *The groups $K_q(C_r^*(G))$ can be computed from the E^4 -page of the spectral sequence, and coincide with*

$$K_q(C_r^*(G)) = \begin{cases} H_1(\mathcal{C}) \oplus \ker(d_{3,0}^3) & \text{if } q \text{ is odd,} \\ \text{coker}(d_{3,0}^3) \oplus H_2(\mathcal{C}) & \text{if } q \text{ is even,} \end{cases}$$

where $d_{3,0}^3 : H_3(\mathcal{C}) \rightarrow H_0(\mathcal{C})$ is the differential appearing on the E^3 -page of the spectral sequence.

Since we are only interested in the rationalized equivariant K -homology, we can actually ignore the presence of any differentials: after tensoring with \mathbb{Q} the Atiyah-Hirzebruch spectral sequence collapses at the E^2 -page [**Lu1**, Remark 3.9]. Thus for q even,

$$K_q(C_r^*(G)) \otimes \mathbb{Q} \cong (E_{0,q}^2 \otimes \mathbb{Q}) \oplus (E_{2,q-2}^2 \otimes \mathbb{Q}) \cong (H_0(\mathcal{C}) \otimes \mathbb{Q}) \oplus (H_2(\mathcal{C}) \otimes \mathbb{Q}),$$

and for q odd,

$$K_q(C_r^*(G)) \otimes \mathbb{Q} \cong (E_{1,q-1}^2 \otimes \mathbb{Q}) \oplus (E_{3,q-3}^2 \otimes \mathbb{Q}) \cong (H_1(\mathcal{C}) \otimes \mathbb{Q}) \oplus (H_3(\mathcal{C}) \otimes \mathbb{Q}).$$

LEMMA 4. *The ranks of the groups $K_q(C_r^*(G)) \otimes \mathbb{Q}$ are given by*

$$\text{rank} \left(K_q(C_r^*(G)) \otimes \mathbb{Q} \right) = \begin{cases} \text{rank} (H_1(\mathcal{C}) \otimes \mathbb{Q}) + \text{rank} (H_3(\mathcal{C}) \otimes \mathbb{Q}) & \text{if } q \text{ is odd,} \\ \text{rank} (H_0(\mathcal{C}) \otimes \mathbb{Q}) + \text{rank} (H_2(\mathcal{C}) \otimes \mathbb{Q}) & \text{if } q \text{ is even.} \end{cases}$$

Remark: Alternatively this result follows directly from the equivariant Chern character being a rational isomorphism [**MV**, Thm. 6.1].

In the next four sections, we explain how to algorithmically compute the ranks of the four groups appearing in Lemma 4.

3.2. 1-skeleton of X/G and the group $H_0(\mathcal{C})$. For the group $H_0(\mathcal{C})$, we make use of the result from [**MV**, Theorem 3.19]. For the convenience of the reader, we restate the theorem:

THEOREM 5. *For G an arbitrary group, we have*

$$H_0(\mathcal{C}) \otimes \mathbb{Q} \cong \mathbb{Q}^{cf(G)},$$

where $cf(G)$ denotes the number of conjugacy classes of elements of finite order in the group G .

This reduces the computation of the rank of $H_0(\mathcal{C}) \otimes \mathbb{Q}$ to finding some algorithm for computing the number $cf(G)$. We now explain how one can compute the integer $cf(G)$ in terms of the 1-skeleton of the space X/G .

For each cell σ in \underline{BG} , we fix a reference cell $\tilde{\sigma} \in \underline{EG}$, having the property that $\tilde{\sigma}$ maps to σ under the quotient map $p : \underline{EG} \rightarrow \underline{BG}$. Associated to each cell σ in \underline{BG} , we have a finite subgroup $G_{\tilde{\sigma}} \leq G$, which is just the stabilizer of the fixed pre-image $\tilde{\sigma} \in \underline{EG}$. Since the stabilizers of two distinct lifts $\tilde{\sigma}, \tilde{\sigma}'$ of the cell σ are conjugate subgroups inside G , we note that the *conjugacy class* of the finite subgroup $G_{\tilde{\sigma}}$ is independent of the choice of lift $\tilde{\sigma}$, and depends solely on the cell $\sigma \in \underline{BG}$. Now given a cell σ in \underline{BG} with a boundary cell τ , we have associated lifts $\tilde{\sigma}, \tilde{\tau}$. Of course, the lift $\tilde{\tau}$ might not lie in the boundary of $\tilde{\sigma}$, but there exists some other lift $\tilde{\tau}'$ of τ which *does* lie in the boundary of $\tilde{\sigma}$. Clearly, we have an inclusion $G_{\tilde{\sigma}} \hookrightarrow G_{\tilde{\tau}'}$. Fix an element $g_{\sigma,\tau} \in G$ with the property that $g_{\sigma,\tau}$ maps the lift $\tilde{\tau}'$ to the lift $\tilde{\tau}$. This gives us a map $\phi_{\tilde{\sigma}}^{\tau} : G_{\tilde{\sigma}} \hookrightarrow G_{\tilde{\tau}}$, obtained by composing the inclusion $G_{\tilde{\sigma}} \hookrightarrow G_{\tilde{\tau}'}$ with the isomorphism $G_{\tilde{\tau}'} \rightarrow G_{\tilde{\tau}}$ given by conjugation by $g_{\sigma,\tau}$. Now the map $\phi_{\tilde{\sigma}}^{\tau}$ isn't well-defined, as there are different possible choices for the element $g_{\sigma,\tau}$. However, if $g'_{\sigma,\tau}$ represents a different choice of element, then since both elements $g_{\sigma,\tau}, g'_{\sigma,\tau}$ map $\tilde{\tau}'$ to $\tilde{\tau}$, we see that the product $(g'_{\sigma,\tau})(g_{\sigma,\tau})^{-1}$ maps $\tilde{\tau}$ to itself, and hence we obtain the equality $g'_{\sigma,\tau} = h \cdot g_{\sigma,\tau}$, where $h \in G_{\tilde{\tau}}$. This implies that the map $\phi_{\tilde{\sigma}}^{\tau}$ is well-defined, up to post-composition by an inner automorphism of $G_{\tilde{\tau}}$.

Consider the set $F(G)$ consisting of the disjoint union of the finite groups $G_{\tilde{v}}$ where v ranges over vertices in the 0-skeleton $(\underline{BG})^{(0)}$ of \underline{BG} . Form the smallest equivalence relation \sim on $F(G)$ with the property that:

- (i) for each vertex $v \in (\underline{BG})^{(0)}$, and elements $g, h \in G_{\tilde{v}}$ which are conjugate within $G_{\tilde{v}}$, we have $g \sim h$, and
- (ii) for each edge $e \in (\underline{BG})^{(1)}$ joining vertices $v, w \in (\underline{BG})^{(0)}$, and element $g \in G_{\tilde{e}}$, we have $\phi_e^v(g) \sim \phi_e^w(g)$.

Note that, although the maps $\phi_{\tilde{\sigma}}^{\tau}$ are not well-defined, the equivalence relation given above *is* well-defined. Indeed, for any given edge $e \in (\underline{BG})^{(1)}$, the maps ϕ_e^v, ϕ_e^w are only well-defined up to inner automorphisms of $G_{\tilde{v}}, G_{\tilde{w}}$. In view of property (i), the resulting property (ii) is independent of the choice of representatives ϕ_e^v, ϕ_e^w .

For a finitely generated group, we let $eq(G)$ denote the number of \sim equivalence classes on the corresponding set $F(G)$. We can now establish:

LEMMA 6. *For G an arbitrary finitely generated group, we have $cf(G) = eq(G)$.*

PROOF. Let us write $g \approx h$ if the elements g, h are conjugate in G . As each element in $F(G)$ is also an element in G , we now have the two equivalence relations \sim, \approx on the set $F(G)$. It is immediate from the definition that $g \sim h$ implies $g \approx h$.

Next, we argue that, for elements $g, h \in F(G)$, $g \approx h$ implies $g \sim h$. To see this, assume that $k \in G$ is a conjugating element, so $g = khk^{-1}$. For the action on \underline{EG} , we know that g, h fix vertices \tilde{v}, \tilde{w} (respectively) in the 0-skeleton $(\underline{EG})^{(0)}$, which project down to vertices $v, w \in (\underline{BG})^{(0)}$ (respectively). Since $g = khk^{-1}$, we also have that g fixes the vertex $k \cdot \tilde{w}$. The g fixed set \underline{EG}^g is contractible, so we can find a path joining \tilde{v} to $k \cdot \tilde{w}$ inside the subcomplex \underline{EG}^g . Within this subcomplex, we can push any path into the 1-skeleton, giving us a sequence of consecutive edges within the graph $(\underline{EG}^g)^{(1)} \subseteq (\underline{EG})^{(1)}$ joining \tilde{v} to $k \cdot \tilde{w}$. This projects down to a path in $(\underline{BG})^{(1)}$ joining the vertex v to the vertex w (as $k \cdot \tilde{w}$ and \tilde{w} lie in the same G -orbit, they have the same projection). Using property (ii), the projected path gives a sequence of elements $g = g_0 \sim g_1 \sim \dots \sim g_k = h$, where each pair g_i, g_{i+1} are in the groups associated to consecutive vertices in the path.

So we now have that the two equivalence relations \sim and \approx coincide on the set $F(G)$, and in particular, have the same number of equivalence classes. Of course, the number of \sim equivalence classes is precisely the number $eq(G)$. On the other hand, any element of finite order g in G must have non-trivial fixed set in \underline{EG} . Since the action is cellular, this forces the existence of a fixed vertex $\bar{v} \in (\underline{EG})^{(0)}$ (which might not be unique). The vertex \bar{v} has an image vertex $v \in (\underline{BG})^{(0)}$ under the quotient map, and hence $g \approx \tilde{g}$ for some element \tilde{g} in the set $F(G)$, corresponding to the subgroup $G_{\bar{v}}$. This implies that the number of \approx equivalence classes in $F(G)$ is equal to $cf(G)$, concluding the proof. \square

Remark: The procedure we described in this section works for any model for \underline{BG} , and would compute the β_0 of the corresponding chain complex. On the other hand, if one has a model for \underline{EG} with the property that the quotient \underline{BG} has few vertices and edges, then it is fairly straightforward to calculate the number $eq(G)$ from the 1-skeleton of \underline{BG} . For the groups we are considering, we can use the model space X . The 1-skeleton of \underline{BG} is then a quotient of the 1-skeleton of the polyhedron P . Along with Lemma 6, this allows us to easily compute the rank of $H_0(\mathcal{C}) \otimes \mathbb{Q}$ for the groups within our class.

3.3. Topology of X/G and the group $H_3(\mathcal{C})$. Our next step is to understand the rank of the group $H_3(\mathcal{C}) \otimes \mathbb{Q}$; this requires an understanding of the differentials appearing in the chain complex \mathcal{C} . In X , if we have a k -cell σ contained in the closure of a $(k+1)$ -cell τ , then we have a natural inclusion of stabilizers $G_\tau \hookrightarrow G_\sigma$. Applying the functor $K_q(C_r^*(-))$, where q is even, we get an induced morphism from the complex representation ring of G_τ to the complex representation ring of G_σ . Concretely, the image of a complex representation ρ of G_τ under this morphism is the induced complex representation $\rho \uparrow := \text{Ind}_{G_\tau}^{G_\sigma} \rho$ in G_σ , with multiplicity given (as usual) by the degree of the attaching map from the boundary sphere $S^k = \partial\tau$ to the sphere $S^k = \sigma/\partial\sigma$. Note that conjugate representations induce up to the same representation.

In the chain complex, the individual terms are indexed by *orbits* of cells in X , rather than individual cells. To see what the chain map does, pick an orbit of $(k+1)$ -cells, and fix an oriented representative τ . Then for each orbit of a k -cell, one can look at the k -cells in that oriented orbit that are incident to τ , call them $\sigma_1, \dots, \sigma_r$. The stabilizer of each of the σ_i is a copy of the same group G_σ (where the identification between these groups is well-defined up to inner automorphisms). For each of these σ_i , the discussion in the previous paragraph allows us to obtain a map on complex representation rings. Finally, one identifies the groups G_{σ_i} with the group G_σ , and take the sum of the maps on the complex representation rings. This completes the description of the chain maps in the complex \mathcal{C} .

Consider a representative σ for the single 3-cell orbit in the G -CW-complex X (we can identify σ with the interior of the polyhedron P). The stabilizer of σ must be trivial (as any element stabilizing σ must stabilize all of X). We conclude that $\mathcal{C}_3 = \bigoplus_{\sigma \in (X/G)^{(3)}} K_q(C_r^*(G_\sigma)) \cong \mathbb{Z}$, and the generator for this group is given by the trivial representation of the trivial group. But inducing up the trivial representation of the trivial group always gives the left regular representation, which is just the sum of all irreducible representations. This tells us that, for each 2-cell in the boundary of σ , the corresponding map on the K -group is non-trivial.

Now when looking at the chain complex, the target of the differential is indexed by *orbits* of 2-cells, rather than individual 2-cells. Each 2-cell orbit has either one or two representatives lying in the boundary of σ . Whether there is one or two can be decided as follows: look at the G -translate σ' of σ which is adjacent to σ across the given boundary 2-cell τ . Since X is a *manifold* model for $\underline{E}G$, there is a unique such σ' . As the stabilizer of the 3-cell is trivial, there is a unique element $g \in G$ which takes σ to σ' . Let τ' denote the pre-image $g^{-1}(\tau)$, a 2-cell in the boundary of σ . Clearly g identifies together the cells τ, τ' in the quotient space X/G .

If $\tau = \tau'$, then the cell τ descends to a boundary cell in quotient space X/G , and the stabilizer of τ is isomorphic to \mathbb{Z}_2 (with non-trivial element given by g). On the other hand, if $\tau \neq \tau'$, then τ descends to an interior cell in the quotient space X/G , with trivial stabilizer.

Now if the 3-cell σ has a boundary 2-cell τ whose stabilizer is \mathbb{Z}_2 , then the orbit of τ intersects the boundary of σ in precisely τ . Looking at the coordinate corresponding to the orbit of τ , we see that in this case the map $\mathbb{Z} \rightarrow \bigoplus_{f \in (X/G)^{(2)}} K_q(C_r^*(G_f))$ in the chain complex is an *injection*, and hence that $E_{3,q}^2 = H_3(\mathcal{C}) = 0$ for all even q .

The other possibility is that *all* boundary 2-cells are pairwise identified, in which case the quotient space X/G is (topologically) a closed manifold. With respect to the induced orientation on the boundary of σ , if any boundary 2-cell τ is identified by an orientation *preserving* pairing to τ' , then the quotient space X/G is a non-orientable manifold. Focusing on the coordinate corresponding to the orbit of τ , we again see that the map $\mathbb{Z} \rightarrow \bigoplus_{f \in (X/G)^{(2)}} K_q(C_r^*(G_f))$ in the chain complex is injective (the generator of \mathbb{Z} maps to ± 2 in the τ -coordinate). So in this case we again conclude that $E_{3,q}^2 = H_3(\mathcal{C}) = 0$ for all even q .

Finally, we have the case where all pairs of boundary 2-cells are identified together using orientation reversing pairings. Then the quotient space X/G is (topologically) a closed *orientable* manifold. In this case, the corresponding map $\mathbb{Z} \rightarrow \bigoplus_{f \in X^{(2)}} K_q(C_r^*(G_f))$ in the chain complex is just the zero map (the generator of \mathbb{Z} maps to 0 in each τ -coordinate, due to the two occurrences with opposite orientations). We summarize our discussion in the following:

LEMMA 7. *For our groups G , the third homology group $H_3(\mathcal{C})$ is either (i) isomorphic to \mathbb{Z} , if the quotient space X/G is topologically a closed orientable manifold, or (ii) trivial in all remaining cases.*

Remark: In [MV, Lemma 3.21], it is shown that the comparison map from $H_i(\mathcal{C})$ to the ordinary homology of the quotient space $H_i(\underline{E}G; \mathbb{Z})$ is an isomorphism in all degrees $i > \dim(\underline{E}G^{\text{sing}}) + 1$, and injective in degree $i = \dim(\underline{E}G^{\text{sing}}) + 1$. Note that most of our Lemma 7 can also be deduced from this result. Indeed, our discussion shows that, in case (i), the singular set is 1-dimensional (i.e. all cells of dimension ≥ 2 have trivial stabilizer), and hence $H_3(\mathcal{C}) \cong H_3(X/G) \cong \mathbb{Z}$. If X/G is non-orientable, then [MV, Lemma 3.21] gives that $H_3(\mathcal{C})$ injects into $H_3(X/G) \cong \mathbb{Z}_2$, so our Lemma provides a bit more information. In the case where X/G has boundary, [MV, Lemma 3.21] implies that $H_3(\mathcal{C})$ injects into $H_3(X/G) \cong 0$, so again recovers our result. We chose to retain our original proof of Lemma 7, as a very similar argument will be subsequently used to calculate $H_2(\mathcal{C})$ (which does not follow from [MV, Lemma 3.21]).

3.4. 2-skeleton of X/G and the rank of $H_2(\mathcal{C})$. Now we turn our attention to the group $H_2(\mathcal{C})$. In order to describe this homology group, we will continue the analysis initiated in the previous section. Recall that we have an explicit (combinatorial) polyhedron P which serves as a fundamental domain for the G -action. We can view the quotient space X/G as obtained from the polyhedron P by identifying together certain faces of P . The CW-structure on X/G is induced from the natural (combinatorial) CW-structure on the polyhedron P . The quotient space X/G inherits the structure of a 3-dimensional orbifold. Note that, if we forget the orbifold structure and just think about the underlying topological space, then X/G is a compact manifold, with possibly non-empty boundary.

There is a close relationship between the *isotropy* of the cells in X/G , thought of as a 3-orbifold, and the *topology* of X/G , viewed as a topological manifold. Indeed, as was discussed in the previous Section 3.3, the stabilizer of any face σ of the polyhedron P is either (i) trivial, or (ii) is isomorphic to \mathbb{Z}_2 . In the first case, there is an element in G which identifies the face σ with some other face of P . So at the level of the quotient space X/G , σ maps to a 2-cell which lies in the *interior* of the closed manifold X/G . In the second case, there are no other faces of the polyhedron P that lie in the G -orbit of σ , and hence σ maps to a boundary 2-cell of X/G . We summarize this analysis in the following

Fact 2: For any 2-cell σ in X/G , we have that:

- i) σ lies in the boundary of X/G if and only if σ has isotropy \mathbb{Z}_2 , and
- ii) σ lies in the interior of X/G if and only if σ has trivial isotropy.

A similar analysis applies to 1-cells. Indeed, the stabilizer of any edge in the polyhedron P must either be (i) a finite cyclic group, or (ii) a finite dihedral group. But case (ii) can only occur if there is some orientation reversing isometry through one of the faces containing the edge. This would force the edge to lie in the boundary of the corresponding face, with the stabilizer of the face being \mathbb{Z}_2 . In view of **Fact 2**, such an edge would have to lie in the boundary of X/G . Conversely, if one has an edge in the boundary of X/G , then it has two adjacent faces (which might actually coincide) in the boundary of X/G , each with stabilizer \mathbb{Z}_2 , given by a reflection in the face. In most cases, these two reflections will determine a dihedral stabilizer for e ; the exception occurs if the two incident faces have stabilizers which coincide in G . In that case, the stabilizer of e will also be a \mathbb{Z}_2 , and will coincide with the stabilizers of the two incident faces. We summarize this discussion as our:

Fact 3: For any 1-cell e in X/G , we have that:

- i) e lies in the interior of X/G if and only if e has isotropy a cyclic group, acting by rotations around the edge,
- ii) if e has isotropy a dihedral group, then e lies in the boundary of X/G ,
- iii) the remaining edges in the boundary of X/G have stabilizer \mathbb{Z}_2 , which coincides with the \mathbb{Z}_2 stabilizer of the incident boundary faces.

With these observations in hand, we are now ready to calculate $H_2(\mathcal{C}) \otimes \mathbb{Q}$. In order to understand this group, we need to understand the kernel of the morphism

$$\Phi : \bigoplus_{\sigma \in (X/G)^{(2)}} K_0(C_r^*(G_\sigma)) \rightarrow \bigoplus_{e \in (X/G)^{(1)}} K_0(C_r^*(G_e)).$$

Indeed, the group $H_2(\mathcal{C})$ is isomorphic to the quotient of $\ker(\Phi)$ by a homomorphic image of $K_0(C_r^*(G_\tau)) \cong \mathbb{Z}$, where τ is a representative for the unique 3-cell orbit. As such, we see that the rank of $H_2(\mathcal{C}) \otimes \mathbb{Q}$ either coincides with the rank of $\ker(\Phi)$, or is one less than the rank of $\ker(\Phi)$.

Our approach to analyzing $\ker(\Phi)$ is to split up this group into smaller pieces, which are more amenable to a geometric analysis. Let us introduce the notation Φ_e , where e is an edge, for the composition of the map Φ with the projection onto the summand $K_0(C_r^*(G_e))$. The next Lemma analyzes the behavior of the map Φ in the vicinity of a boundary edge with stabilizer a dihedral group.

LEMMA 8. *Let e be a boundary edge, with stabilizer a dihedral group D_n . Then we have:*

(i) *if σ is an incident interior face, then*

$$\Phi_e\left(K_0(C_r^*(G_\sigma))\right) \subseteq \mathbb{Z} \cdot \langle 1, 1, \dots, 1, 1 \rangle \leq K_0(C_r^*(D_n)),$$

(ii) *if σ_1, σ_2 are the incident boundary faces, then*

$$\Phi_e\left(K_0(C_r^*(G_{\sigma_1})) \oplus K_0(C_r^*(G_{\sigma_2}))\right) \cap \mathbb{Z} \cdot \langle 1, 1, \dots, 1, 1 \rangle = \langle 0, \dots, 0 \rangle.$$

Note that Lemma 8 tells us that, from the viewpoint of finding elements in $\ker(\Phi)$, boundary faces and interior faces that come together along an edge with dihedral stabilizer have *no interactions*.

PROOF. There are precisely *two* boundary faces which are incident to e , and some indeterminate number of interior faces which are incident to e . From **Fact 2**, the boundary faces each have corresponding $G_\sigma \cong \mathbb{Z}_2$, while the interior faces each have $G_\sigma \cong 1$. For the boundary faces, we have

$$K_0(C_r^*(G_\sigma)) = K_0(C_r^*(\mathbb{Z}_2)) \cong \mathbb{Z} \oplus \mathbb{Z}$$

with generators given by the trivial representation and the sign representation of the group \mathbb{Z}_2 . The interior faces have $K_0(C_r^*(G_\sigma)) \cong \mathbb{Z}$, generated by the trivial representation of the trivial group.

For each incidence of σ on e , the effect of Φ_e on the generator is obtained by inducing up representations. But the trivial representation of the trivial group always induces up to the left regular representation on the ambient group. The latter is the sum of all irreducible representations, hence corresponds to the element $\langle 1, \dots, 1 \rangle \leq K_0(C_r^*(\mathbb{Z}_n))$. This tells us that, for each internal face, the image of Φ_e lies in the subgroup $\mathbb{Z} \cdot \langle 1, 1, \dots, 1, 1 \rangle$, establishing (i).

On the other hand, an easy calculation (see Appendix A) shows that, if $\sigma_1, \sigma_2 \in (X/G)^{(2)}$ are the two boundary faces incident to e , then in the e -coordinate we have

$$\Phi_e\left(K_0(C_r^*(G_{\sigma_1})) \oplus K_0(C_r^*(G_{\sigma_2}))\right) \cap \mathbb{Z} \cdot \langle 1, 1, \dots, 1, 1 \rangle = \langle 0, \dots, 0 \rangle,$$

which is the statement of (ii). □

To analyze $\ker(\Phi)$, we need to introduce some auxiliary spaces. Recall that X/G is topologically a closed 3-manifold, possibly with boundary. We introduce the following terminology for boundary components:

- a boundary component is *dihedral* if it has no edges with stabilizer \mathbb{Z}_2 (i.e. all its edges have stabilizers which are dihedral groups),

- a boundary component is *non-dihedral* if it is not dihedral (i.e. it contains at least one edge with stabilizer \mathbb{Z}_2),
- a dihedral boundary component is *odd* if it has no edges with stabilizer of the form D_{2k} (i.e. all its edges have stabilizers of the form D_{2k+1}), and
- a dihedral boundary component is *non-odd* if it contains an edge e with stabilizer of the form D_{2k} (i.e. an edge whose stabilizer has order a multiple of 4).

Let s denote the number of orientable non-odd dihedral boundary components, and let t denote the number of orientable odd dihedral boundary components. Note that it is straightforward to calculate the integers s, t from the polyhedral fundamental domain P for the G -action on X .

Next, form the 2-complex Y by taking the union of the closure of all interior faces of X/G , along with all the non-dihedral boundary components. We denote by $\partial Y \subset Y$ the subcomplex consisting of all non-dihedral boundary components. By construction, ∂Y consists precisely of the subcomplex generated by the 2-cells in $Y \cap \partial(X/G)$, so the choice of notation should cause no confusion. Let Z denote the union of all dihedral boundary components of X/G .

By construction, every 2-cell in X/G appears either in Y or in Z , but not in both. This gives rise to a decomposition of the indexing set $(X/G)^{(2)} = Y^{(2)} \amalg Z^{(2)}$, which in turn yields a splitting:

$$\bigoplus_{\sigma \in (X/G)^{(2)}} K_0(C_r^*(G_\sigma)) = \left[\bigoplus_{\sigma \in Y^{(2)}} K_0(C_r^*(G_\sigma)) \right] \oplus \left[\bigoplus_{\sigma \in Z^{(2)}} K_0(C_r^*(G_\sigma)) \right].$$

Let us denote by Φ_Y and Φ_Z the restrictions of Φ to the first and second summands described above. We then have the following:

LEMMA 9. *There is a splitting $\ker(\Phi) = \ker(\Phi_Y) \oplus \ker(\Phi_Z)$.*

PROOF. We clearly have the inclusion $\ker(\Phi_1) \oplus \ker(\Phi_2) \subseteq \ker(\Phi)$, so let us focus on the opposite containment. If we have some arbitrary element $v \in \ker(\Phi)$, we can decompose $v = v_Y + v_Z$, where we have $v_Y \in \bigoplus_{\sigma \in Y^{(2)}} K_0(C_r^*(G_\sigma))$, and $v_Z \in \bigoplus_{\sigma \in Z^{(2)}} K_0(C_r^*(G_\sigma))$. Let us first argue that $v_Z \in \ker(\Phi_Z)$, i.e. that $\Phi(v_Z) = 0$. This is of course equivalent to showing that for every edge e , we have $\Phi_e(v_Z) = 0$.

Since v_Z is supported on 2-cells lying in Z , it is clear that for any edge $e \notin Z$, we have $\Phi_e(v_Z) = 0$. For edges $e \subset Z$, we have:

$$0 = \Phi_e(v) = \Phi_e(v_Y + v_Z) = \Phi_e(v_Y) + \Phi_e(v_Z).$$

This tells us that $\Phi_e(v_Z) = \Phi_e(-v_Y)$ lies in the intersection

$$(2) \quad \Phi_e \left(\bigoplus_{\sigma \in Y^{(2)}} K_0(C_r^*(G_\sigma)) \right) \cap \Phi_e \left(\bigoplus_{\sigma \in Z^{(2)}} K_0(C_r^*(G_\sigma)) \right).$$

But $Y^{(2)}$ contains all the *interior* faces incident to e , while $Z^{(2)}$ contains all *boundary* faces incident to e . Since $e \subset Z$, and Z is the union of all dihedral boundary components of X/G , we have that the stabilizer G_e must be dihedral. Applying Lemma 8, we see that the intersection in equation (2) consists of just the zero vector, and hence $\Phi_e(v_Z) = 0$.

Since we have shown that $\Phi_e(v_Z) = 0$ holds for all edges e , we obtain that $v_Z \in \ker(\Phi_Z)$, as desired. Finally, we have that

$$\Phi(v_Y) = \Phi(v - v_Z) = \Phi(v) - \Phi(v_Z) = 0$$

as both v, v_Z are in the kernel of Φ . We conclude that $v_Y \in \ker(\Phi_Y)$, concluding the proof of the Lemma. \square

We now proceed to analyze each of $\ker(\Phi_Y), \ker(\Phi_Z)$ separately. We start with:

LEMMA 10. *The group $\ker(\Phi_Z)$ is free abelian, of rank equal to $s + 2t$.*

Before establishing Lemma 10, recall that s, t counts the number of orientable dihedral boundary components of X/G which are non-odd and odd, respectively. From the definition of Z , we see that the number of connected components of the space Z is precisely $s + t$.

PROOF. It is obvious that $\ker(\Phi_Z)$ decomposes as a direct sum of the kernels of Φ restricted to the individual connected components of Z , which are precisely the dihedral boundary components of X/G . So we can argue one dihedral boundary component at a time. On a fixed dihedral boundary component, we have that each 2-cell contributes a $\mathbb{Z} \oplus \mathbb{Z}$ to the source of the map Φ , with canonical (ordered) basis given by the trivial representation and the sign representation on \mathbb{Z}_2 . Fix a boundary edge e , and let σ_1, σ_2 be the two boundary faces incident to e . We assume that the two faces are equipped with compatible orientations, and let (a_i, b_i) be elements in the groups $K_0(C_r^*(G_{\sigma_i})) \cong \mathbb{Z} \oplus \mathbb{Z}$. Now assume that $\Phi_e((a_1, b_1 \mid a_2, b_2)) = 0$. Then an easy computation (see Appendix A) shows that:

- a) if e has stabilizer of the form D_{2k+1} , then we must have $a_1 = a_2$ and $b_1 = b_2$,
- b) if e has stabilizer of the form D_{2k} , then we must have $a_1 = a_2 = b_1 = b_2$

(and since Z consists of dihedral boundary components, there are no edges e in Z with stabilizer \mathbb{Z}_2). Note that reversing the orientation on one of the faces just changes the sign of the corresponding entries. We can now calculate the contribution of each boundary component to $\ker(\Phi_Z)$.

Non-orientable components: Any such boundary component contains an embedded Möbius band. Without loss of generality, we can assume that the sequence of faces $\sigma_1, \dots, \sigma_r$ cyclically encountered by this Möbius band are all distinct. At the cost of flipping the orientations on $\sigma_i, 2 \leq i \leq r$, we can assume that consecutive pairs are coherently oriented. Since we have a Möbius band, this forces the orientations of σ_1 and σ_r to be non-coherent along their common edge. So if we have an element lying in $\ker(\Phi_Z)$, the coefficients along the cyclic sequence of faces must satisfy (regardless of the edge stabilizers):

$$a_1 = a_2 = \dots = a_k = -a_1$$

$$b_1 = b_2 = \dots = b_k = -b_1$$

This forces $a_1 = b_1 = 0$. Regardless of the orientations and edge stabilizers, equations (a) and (b) imply that this propagates to force all coefficients to equal zero. We conclude that any element in $\ker(\Phi_Z)$ must have all zero coefficients in the 2-cells corresponding to any non-orientable boundary component.

Orientable odd components: Fix a coherent orientation of all the 2-cells in the boundary component. Then in view of equation (a) above, elements lying in $\ker(\Phi)$ must have all a_i -coordinates equal, and all b_i -coordinates equal (as one ranges over 2-cells within this fixed boundary component). This gives two degrees of freedom, and hence such a boundary component contributes a \mathbb{Z}^2 to $\ker(\Phi_Z)$.

Orientable non-odd components: Again, let us fix a coherent orientation of all the 2-cells in the boundary component. As in the odd component case, any element in $\ker(\Phi_Z)$ must have all a_i -coordinates equal, and all b_i -coordinates equal. However, the presence of a single edge with stabilizer of the form D_{2k} forces, for the two adjacent faces, to have corresponding a - and b -coordinates equal (see equation (b) above). This in turn propagates to yield that *all* the a - and b -coordinates must be equal. As such, we have one degree of freedom for elements in the kernel, and hence such a boundary component contributes a single \mathbb{Z} to $\ker(\Phi_Z)$. This concludes the proof of Lemma 10. \square

Next we focus on the group $\ker(\Phi_Y)$. We would like to relate $\ker(\Phi_Y)$ with the second homology of the space Y . Let \mathcal{A} denote the cellular chain complex for the CW-complex Y , and let $d_Y : \mathcal{A}_2 \rightarrow \mathcal{A}_1$ denote the differentials in the cellular chain complex. Since Y is a 2-dimensional CW-complex, we have that $H_2(Y) = \ker(d_Y)$. Our next step is to establish:

LEMMA 11. *There is a split surjection $\phi : \ker(\Phi_Y) \rightarrow \ker(d_Y)$, providing a direct sum decomposition $\ker(\Phi_Y) \cong \ker(\phi) \oplus \ker(d_Y)$.*

PROOF. Let $\mathcal{D} \subset \mathcal{C}$ denote the subcomplex of our original chain complex determined by the subcollection of indices $Y^{(k)} \subset (X/G)^{(k)}$. By construction, the map Φ_Y we are interested in is the boundary operator $\Phi_Y : \mathcal{D}_2 \rightarrow \mathcal{D}_1$ appearing in the chain complex \mathcal{D} . We define the map

$$\hat{\phi} : \mathcal{D}_2 = \bigoplus_{\sigma \in Y^{(2)}} K_0(C_r^*(G_\sigma)) \rightarrow \bigoplus_{\sigma \in Y^{(2)}} \mathbb{Z} = \mathcal{A}_2$$

as the direct sum of maps $\hat{\phi}_\sigma : K_0(C_r^*(G_\sigma)) \rightarrow \mathbb{Z}$, where:

- if G_σ is trivial, then $\hat{\phi}_\sigma : \mathbb{Z} \rightarrow \mathbb{Z}$ takes the generator for $K_0(C_r^*(G_\sigma)) = \mathbb{Z}$ given by the trivial representation to the element $1 \in \mathbb{Z}$, and
- if $G_\sigma = \mathbb{Z}_2$, then $\hat{\phi}_\sigma : \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z}$ is given by $\hat{\phi}_\sigma(\langle 1, 0 \rangle) = 1$, $\hat{\phi}_\sigma(\langle 0, 1 \rangle) = 0$, where, as usual, $\langle 1, 0 \rangle, \langle 0, 1 \rangle$ correspond to the trivial representation and the sign representation respectively.

For any element $z \in \ker(\Phi_Y)$, a computation shows that $(d_Y \circ \hat{\phi})(z) = 0$, and hence $\hat{\phi}$ restricts to a morphism $\phi : \ker(\Phi_Y) \rightarrow \ker(d_Y)$.

Next, we argue that the map $\phi : \ker(\Phi_Y) \rightarrow \ker(d_Y)$ is surjective. To see this, we construct a map $\bar{\phi} : \mathcal{A}_2 \rightarrow \mathcal{D}_2$ as a direct sum of maps $\bar{\phi}_\sigma : \mathbb{Z} \rightarrow K_0(C_r^*(G_\sigma))$. In terms of our usual generating sets for the groups $K_0(C_r^*(G_\sigma))$, the maps $\bar{\phi}_\sigma$ are given by:

- if G_σ is trivial, then $\bar{\phi}_\sigma : \mathbb{Z} \rightarrow \mathbb{Z}$ is defined by $\bar{\phi}_\sigma(1) = 1$, and
- if $G_\sigma = \mathbb{Z}_2$, then $\bar{\phi}_\sigma : \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$ is defined by $\bar{\phi}_\sigma(1) = \langle 1, 1 \rangle$.

We clearly have that $\hat{\phi} \circ \bar{\phi} : \mathcal{A}_2 \rightarrow \mathcal{A}_2$ is the identity, and an easy computation shows that if $z \in \ker(d_Y)$, then $\bar{\phi}(z) \in \ker(\Phi_Y)$. We conclude that the restriction $\phi : \ker(\Phi_Y) \rightarrow \ker(d_Y)$ is surjective, and that the restriction of $\bar{\phi}$ to $\ker(d_Y)$ provides a splitting of this surjection. Since the map ϕ is a split surjection, we see that $\ker(\Phi_Y) \cong \ker(d_Y) \oplus \ker(\phi)$, completing the proof of Lemma 11. \square

So the last step is to identify $\ker(\phi)$. Recall that Y is a 2-complex which contains, as a subcomplex, the union of all boundary components of X/G which have an edge with stabilizer \mathbb{Z}_2 . This subcomplex was denoted by $\partial Y \subset Y$. We

can again call a connected component in ∂Y *odd* if all its edges have stabilizers of the form D_{2k+1} , and *non-odd* otherwise (i.e. some edge has stabilizer of the form D_{2k}). Let t' denote the number of orientable, odd connected components in ∂Y . Then we have:

LEMMA 12. *The group $\ker(\phi)$ is free abelian, of rank $= t'$.*

PROOF. From the definition of ϕ , it is easy to see what form an element in $\ker(\phi)$ must have: in terms of the splitting $\mathcal{D}_2 = \bigoplus_{\sigma \in Y^{(2)}} K_0(C_r^*(G_\sigma))$, the element can only have non-zero terms in the coordinates corresponding to 2-cells in ∂Y . Moreover, in the coordinates $\sigma \in (\partial Y)^{(2)}$, the entries in the corresponding $K_0(C_r^*(G_\sigma)) \cong \mathbb{Z} \oplus \mathbb{Z}$ must lie in the subgroup $\mathbb{Z} \cdot \langle 0, 1 \rangle$. Finally, the fact that the elements we are considering lie in $\ker(\Phi_Y)$ means that, at each edge $e \in (\partial Y)^{(1)}$, with incident edges σ_1, σ_2 , we must have that the corresponding coefficients $\langle 0, b_1 \rangle \in K_0(C_r^*(G_{\sigma_1}))$ and $\langle 0, b_2 \rangle \in K_0(C_r^*(G_{\sigma_2}))$ sum up to zero, i.e. that $b_1 + b_2 = 0$. These properties almost characterize elements in $\ker(\phi)$. Clearly, we can again analyze the situation one connected component of ∂Y at a time. As in the argument for Lemma 10, there are cases to consider:

Non-odd component: In the case where an element $z \in \ker(\phi)$ is supported entirely on a non-odd boundary component, there is one additional constraint. For the two faces σ_1, σ_2 incident to the edge with stabilizer D_{2k} , the fact that $z \in \ker(\Phi)$ forces the corresponding coefficients to satisfy $b_1 = b_2 = a_1 = a_2$ (see equation (b) in the proof of Lemma 10). Since $z \in \ker(\phi)$, we also have $a_1 = a_2 = 0$. This implies that the coefficients $b_1 = b_2$ must also vanish. But then all the b_i coefficients must vanish. We conclude that any element $z \in \ker(\phi)$ must have zero coefficients on all 2-cells contained in a non-odd component.

Odd component: In the case where an element $z \in \ker(\phi)$ is supported entirely on an odd boundary component, the conditions discussed above actually do characterize an element in $\ker(\phi)$. This is due to the fact that, at every edge, the b_i components are actually independent of the a_i components (see equation (a) in the proof of Lemma 10). But the description given above is just stating that the b_i form the coefficients for an (ordinary) 2-cycle in the boundary component. Such a 2-cycle can only exist if the boundary component is orientable, in which case there is a 1-dimensional family of such 2-cycles. We conclude that the orientable, odd components each contribute a \mathbb{Z} to $\ker(\phi)$, while the non-orientable odd components make no contributions.

Since t' is the number of orientable, odd components in ∂Y , the Lemma follows. \square

We now have all the required ingredients to establish:

THEOREM 13. *The group $\ker(\Phi)$ is free abelian of rank $s + t' + 2t + \beta_2(Y)$.*

PROOF. Lemma 9 provides us with a splitting $\ker(\Phi) = \ker(\Phi_Y) \oplus \ker(\Phi_Z)$. Lemma 10 shows that $\ker(\Phi_Z)$ is free abelian of rank $= s + 2t$. Lemma 11 yields the splitting $\ker(\Phi_Y) \cong \ker(\phi) \oplus \ker(d_Y)$. Finally, Lemma 12 tells us that $\ker(\phi)$ is free abelian of rank $= t'$, while the fact that Y is a 2-complex tells us that $\ker(d_Y)$ is free abelian of rank $= \beta_2(Y)$. \square

As a consequence, we obtain the desired formula for $\beta_2(\mathcal{C})$.

COROLLARY 14. *For our groups G , we have that the rank of $H_2(\mathcal{C}) \otimes \mathbb{Q}$ is either:*

- $\beta_2(Y)$ if X/G is a closed, oriented, 3-manifold, or
- $s + t' + 2t + \beta_2(Y) - 1$ otherwise.

Remark: Corollary 14 gives us an algorithmically efficient method for computing $\beta_2(\mathcal{C})$, as it merely requires counting certain boundary components of X/G (to determine the integers s, t, t'), along with the calculation of the second Betti number of an explicit 2-complex (for the $\beta_2(Y)$ term).

3.5. Euler characteristic and the rank of $H_1(\mathcal{C})$. Using the procedure described in the previous section, we will now assume that the ranks $\beta_0(\mathcal{C}), \beta_2(\mathcal{C})$, and $\beta_3(\mathcal{C})$ have already been calculated. In order to compute the rank of $H_1(\mathcal{C}) \otimes \mathbb{Q}$, we recall that any chain complex has an associated *Euler characteristic*. The latter is defined to be the alternating sum of the ranks of the groups appearing in the chain complex. It is an elementary exercise to verify that the Euler characteristic also coincides with the alternating sum of the ranks of the homology groups of the chain complex.

In our specific case, the Euler characteristic $\chi(\mathcal{C})$ of the chain complex \mathcal{C} can easily be calculated from the various groups G_σ , where σ ranges over the cells in \underline{BG} . Each cell σ in \underline{BG} contributes $(-1)^{\dim \sigma} c(G_\sigma)$, where $c(G_\sigma)$ is the number of conjugacy classes in the stabilizer G_σ of the cell. Since the homology groups $H_i(\mathcal{C})$ vanish when $i \neq 0, 1, 2, 3$, we also have the alternate formula

$$\chi(\mathcal{C}) = \beta_0(\mathcal{C}) - \beta_1(\mathcal{C}) + \beta_2(\mathcal{C}) - \beta_3(\mathcal{C})$$

This allows us to solve for the rank of $H_1(\mathcal{C}) \otimes \mathbb{Q}$, yielding

LEMMA 15. *For our groups G , we have that the rank of $H_1(\mathcal{C}) \otimes \mathbb{Q}$ coincides with $\beta_1(\mathcal{C}) = \beta_0(\mathcal{C}) + \beta_2(\mathcal{C}) - \beta_3(\mathcal{C}) - \chi(\mathcal{C})$.*

4. Some examples

We illustrate our algorithm by computing the rational topological K -theory of several groups. The first two examples are classes of groups for which the topological K -theory has already been computed. Since our algorithm does indeed recover (rationally) the same results, these examples serve as a check on our method. The last three examples provide some new computations.

The first example considers the particular case where G is additionally assumed to be torsion-free. As a concrete special case, we deal with any semi-direct product of \mathbb{Z}^2 with \mathbb{Z} (the integral computation for these groups can be found in the recent thesis of Isely [I]). The second example considers a finite extension of the integral Heisenberg group by \mathbb{Z}_4 . The *integral* topological K -theory (and algebraic K - and L -theory) for this group has already been computed by Lück [Lu3].

The third and fourth classes of examples are hyperbolic Coxeter groups that have previously been considered by Lafont, Ortiz, and Magurn in [LOM, Example 7], and [LOM, Example 8] respectively (where their lower algebraic K -theory was computed). The fifth example is an affine split crystallographic group, whose algebraic K -theory has been studied by Farley and Ortiz [FO].

4.1. Torsion-free examples. In the special case where G is *torsion-free*, our algorithm becomes particularly simple, as we now proceed to explain.

Let G be a torsion-free group with a cocompact, 3-manifold model X for the classifying space $\underline{E}G = EG$. Firstly, recall that $\beta_0(\mathcal{C}) = cf(G)$, where $cf(G)$ denotes the number of conjugacy classes of elements of finite order in G (our Lemma 6 provides a way of computing this integer from the 1-skeleton of X/G). Since G is torsion-free, we obtain that $\beta_0(\mathcal{C}) = 1$.

Next, we consider the orbit space $M := X/G$. Recall that any boundary component in the 3-manifold M gives 2-cells with stabilizer \mathbb{Z}_2 . Since G is torsion-free, the orbit space M has no boundary, hence is a *closed* 3-manifold. Then Lemma 7 tells us that

$$\beta_3(\mathcal{C}) = \begin{cases} 1 & \text{if } M \text{ orientable,} \\ 0 & \text{if } M \text{ non-orientable.} \end{cases}$$

To compute $\beta_2(\mathcal{C})$ we apply Corollary 14. The 2-simplex Y is just the 2-skeleton of M and, as $\partial M = \emptyset$, we obtain that

$$\beta_2(\mathcal{C}) = \begin{cases} \beta_2(Y) & \text{if } M \text{ orientable,} \\ \beta_2(Y) - 1 & \text{if } M \text{ non-orientable.} \end{cases}$$

Note that the 2nd Betti number of $Y = M^{(2)}$ can be deduced from that of M , as follows. Since M is obtained from Y by attaching a single 3-cell, the Mayer-Vietoris exact sequence gives

$$0 \longrightarrow H_3(M) \xrightarrow{c} H_2(S^2) \xrightarrow{g} H_2(Y) \oplus H_2(\mathbb{D}^3) \twoheadrightarrow H_2(M) \longrightarrow 0$$

(Here \mathbb{D}^3 is the attaching 3-disk.) Recall that $H_2(S^2) \cong \mathbb{Z}$ and $H_2(\mathbb{D}^3) = 0$. Hence if M is orientable, $H_3(M) \cong \mathbb{Z}$, the image of the map g is then torsion and tensoring with \mathbb{Q} gives $\beta_2(Y) = \beta_2(M)$. If M is non-orientable, $H_3(M) = 0$, the map g is injective and we have $\beta_2(Y) - 1 = \beta_2(M)$. Hence in all cases we actually obtain that $\beta_2(\mathcal{C}) = \beta_2(M)$.

To compute $\beta_1(\mathcal{C})$ we should find $\chi(\mathcal{C})$. Since G is torsion-free all the isotropy groups are trivial and thus $\chi(\mathcal{C}) = \chi(M)$. Since M is a closed 3-manifold, $\chi(M)$ and therefore $\chi(\mathcal{C})$ are zero. Finally, Lemma 15 gives

$$\beta_1(\mathcal{C}) = \beta_0(\mathcal{C}) + \beta_2(\mathcal{C}) - \beta_3(\mathcal{C}) - \chi(\mathcal{C}) = \beta_2(M) - \beta_3(\mathcal{C}) + 1,$$

which simplifies to two cases:

$$\beta_1(\mathcal{C}) = \begin{cases} \beta_2(M) & \text{if } M \text{ is orientable,} \\ \beta_2(M) + 1 & \text{if } M \text{ is not orientable.} \end{cases}$$

Finally applying Lemma 4, we deduce the:

COROLLARY 16. *Let G be a torsion-free group, and X be a cocompact 3-manifold model for $\underline{E}G = EG$. Assume that the quotient 3-manifold $M = X/G$ is geometrizable (this is automatic, for instance, if M is orientable). Then we have that*

$$\text{rank}(K_q(C_r^*(G)) \otimes \mathbb{Q}) = \beta_2(M) + 1$$

holds for all q .

Remark: The number above is the sum of the even-dimensional Betti numbers of M (which coincides with the sum of the odd-dimensional Betti numbers of M , by Poincaré duality) — compare this with the Remark after Lemma 4.

Remark: Note that for G torsion-free, the dimension of the singular part is -1 and hence Lemma 3.21 in [MV] gives $H_i(\mathcal{C}) \cong H_i(M)$ for $i > 0$ and an injection $H_0(\mathcal{C}) \hookrightarrow H_0(M)$. From this it follows that $\beta_i(\mathcal{C}) = \beta_i(M)$ for $i = 1, 2, 3$ and $\beta_0(\mathcal{C}) = \beta_0(M)$ since $1 \leq \beta_0(\mathcal{C}) \leq \beta_0(M) = 1$. This is shown above by direct application of our algorithm.

Semi-direct product of \mathbb{Z}^2 and \mathbb{Z} . For a concrete example of the torsion-free case, consider a semi-direct product $G_\alpha = \mathbb{Z}^2 \rtimes_\alpha \mathbb{Z}$, where $\alpha \in \text{Aut}(\mathbb{Z}^2) = GL_2(\mathbb{Z})$. The automorphism α can be realized (at the level of the fundamental group) by an affine self diffeomorphism of the 2-torus $T^2 = S^1 \times S^1$, $f: T^2 \rightarrow T^2$. The mapping torus M_f of the map f yields a closed 3-manifold which is aspherical and satisfies $\pi_1(M_f) \cong G_\alpha$. Hence it is a model of BG_α and its universal cover a model of EG_α . Since G_α is torsion-free (as it is the semi-direct product of torsion-free groups), these spaces are also models of $\underline{B}G_\alpha$ respectively $\underline{E}G_\alpha$. In particular, these examples fall under the purview of Corollary 16, telling us that $\text{rank}(K_q(C_r^*(G_\alpha)) \otimes \mathbb{Q}) = \beta_2(M_f) + 1$. To complete the calculation, we just need to compute the 2nd Betti number of the 3-manifold M_f . This follows from a straightforward application of the Leray-Serre spectral sequence. We have included the details in Appendix B and here we only quote the result

$$\beta_2(M_f) = \begin{cases} 3 & \text{if } \alpha = \text{Id}, \\ 2 & \text{if } \det(\alpha) = 1, \text{tr}(\alpha) = 2, \alpha \neq \text{Id}, \\ 1 & \text{if } \det(\alpha) = 1, \text{tr}(\alpha) \neq 2, \\ 1 & \text{if } \det(\alpha) = -1, \text{tr}(\alpha) = 0, \\ 0 & \text{if } \det(\alpha) = -1, \text{tr}(\alpha) \neq 0. \end{cases}$$

Adding 1 we obtain

$$K_q(C_r^*(G_\alpha)) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q}^4 & \text{if } \alpha = \text{Id}, \\ \mathbb{Q}^3 & \text{if } \det(\alpha) = 1, \text{tr}(\alpha) = 2, \alpha \neq \text{Id}, \\ \mathbb{Q}^2 & \text{if } \det(\alpha) = 1, \text{tr}(\alpha) \neq 2, \\ \mathbb{Q}^2 & \text{if } \det(\alpha) = -1, \text{tr}(\alpha) = 0, \\ \mathbb{Q} & \text{if } \det(\alpha) = -1, \text{tr}(\alpha) \neq 0. \end{cases}$$

These results agree with the integral computations in Isely's thesis [I, pp. 5-7], giving us a first check on our method.

4.2. Nilmanifold example. In the previous section, we discussed examples where the group was torsion-free, and hence the quotient space was a closed 3-manifold. In this next example, we have a group *with* torsion, but with quotient space again a closed 3-manifold.

The real Heisenberg group $\text{Hei}(\mathbb{R})$ is the Lie group of upper unitriangular, 3×3 matrices with real entries. It is naturally homeomorphic to \mathbb{R}^3 . The integral Heisenberg group $\text{Hei}(\mathbb{Z})$ is the discrete subgroup consisting of matrices whose entries are

in \mathbb{Z} . There is an automorphism $\sigma \in \text{Aut}(\text{Hei}(\mathbb{R}))$ of order 4 given by:

$$\sigma : \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -y & z - xy \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}.$$

This automorphism restricts to an automorphism of the discrete subgroup $\text{Hei}(\mathbb{Z})$, allowing us to define the group $G := \text{Hei}(\mathbb{Z}) \rtimes \mathbb{Z}_4$. An explicit presentation of the group G is given by

$$G := \left\langle a, b, c, t \mid \begin{array}{l} [a, c] = [b, c] = 1, \quad [a, b] = c, \quad t^4 = 1 \\ tat^{-1} = b, \quad tbt^{-1} = a^{-1}, \quad tct^{-1} = c \end{array} \right\rangle$$

where as usual, $[x, y]$ denotes the commutator of the elements x, y . In the above presentation, we are identifying the generators a, b, c with the matrices in $\text{Hei}(\mathbb{Z})$ given by

$$T_a = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T_b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad T_c = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

These generate the normal subgroup $\text{Hei}(\mathbb{Z}) \triangleleft G$, while the conjugation by the last generator t acts via the automorphism $\sigma \in \text{Aut}(\text{Hei}(\mathbb{Z}))$.

The action of $\text{Hei}(\mathbb{Z})$ on $\text{Hei}(\mathbb{R})$ given by left multiplication and the action of \mathbb{Z}_4 on $\text{Hei}(\mathbb{R})$ given by the automorphism σ fit together to give an action of the group G on $\text{Hei}(\mathbb{R})$. It is shown in [Lu3, Lemma 2.4] that this action on $\text{Hei}(\mathbb{R})$ provides a cocompact model for \underline{EG} , with orbit space $G \backslash \underline{EG}$ homeomorphic to S^3 . In order to apply our algorithm, we need to identify a G -CW-structure on $\text{Hei}(\mathbb{R})$. Let us identify \mathbb{R}^3 with $\text{Hei}(\mathbb{R})$ via the map

$$(x, y, z) \leftrightarrow \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Via this identification, we will think of G as acting on \mathbb{R}^3 .

The action of the index four subgroup $\text{Hei}(\mathbb{Z}) \triangleleft G$ on \mathbb{R}^3

$$(n, m, l) \cdot (x, y, z) = (x + n, y + m, z + ny + l)$$

is free. The quotient space $\text{Hei}(\mathbb{Z}) \backslash \mathbb{R}^3$ can be identified in two steps. First, we quotient out by the normal subgroup $H := \langle T_b, T_c \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$. On any hyperplane given by fixing the x -coordinate $x = x_0$, the subgroup H leaves the hyperplane invariant, with the generators T_b, T_c translating by one in the y and z coordinates respectively. Quotienting out by H , we obtain that $H \backslash \mathbb{R}^3$ is homeomorphic to $\mathbb{R} \times T^2$, where the T^2 refers to the standard torus obtained from the unit square (centered at the origin) by identifying the opposite sides. The quotient $\text{Hei}(\mathbb{Z}) \backslash \mathbb{R}^3$ can now be identified by looking at the action of the quotient group $\text{Hei}(\mathbb{Z})/H$ on the space $\mathbb{R} \times T^2$. The generator for $\mathbb{Z} \cong \text{Hei}(\mathbb{Z})/H$, being the image of the matrix $T_x \in \text{Hei}(\mathbb{Z})$, acts by $(x, y, z) \mapsto (x + 1, y, z + y)$. Putting this together, we see that a fundamental domain for the $\text{Hei}(\mathbb{Z})$ -action on \mathbb{R}^3 is given by the unit cube $[-1/2, 1/2]^3$ centered at the origin. The quotient 3-manifold $M := \text{Hei}(\mathbb{Z}) \backslash \mathbb{R}^3$ can now be obtained from the cube via a suitable identification of the faces. The manifold M can also be thought of as the mapping torus of the map $\phi : T^2 \rightarrow T^2$ given by $(y, z) \mapsto (y, y + z) \pmod{1}$.

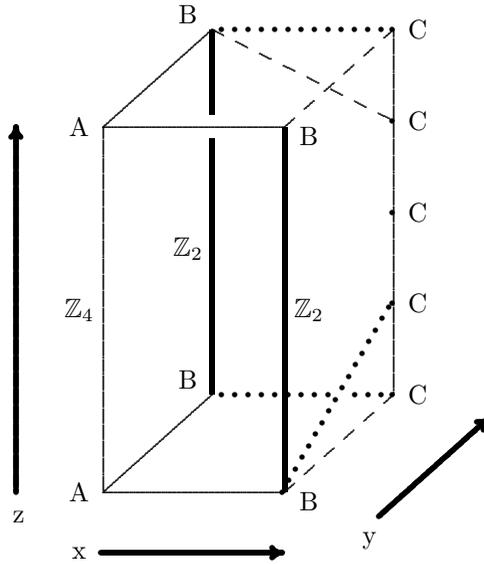


FIGURE 1. $P = [0, \frac{1}{2}] \times [0, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ is a fundamental polyhedron for the action of G on \mathbb{R}^3 . In the quotient space $G \backslash \mathbb{R}^3$, vertices with the same label are identified, as are edges with the same shading. The four edges with both endpoints labelled C are identified with the upward orientation. All faces with the same labels are identified in quotient space. Three edges (two in the quotient) have non-trivial isotropy, as indicated.

Next, we identify a fundamental domain for the G -action on \mathbb{R}^3 . Observe that, since $\text{Hei}(\mathbb{Z}) \triangleleft G$, there is an induced $G/\text{Hei}(\mathbb{Z}) \cong \mathbb{Z}_4$ on M , and a natural identification between $G \backslash \mathbb{R}^3$ and $\mathbb{Z}_4 \backslash M$. The manifold M naturally fibers over T^2 , with fiber S^1 , via the projection onto the (x, y) -plane. The \mathbb{Z}_4 action preserves the S^1 -fibers, so induces an action on the 2-torus T^2 . At the level of the fundamental domain $[-1/2, 1/2]^2 \subset \mathbb{R}^2$ in the (x, y) -plane, the \mathbb{Z}_4 -action is given by $(x, y) \mapsto (-y, x)$. This tells us that a fundamental domain for the \mathbb{Z}_4 -action can be obtained by restricting to the square $[0, 1/2] \times [0, 1/2]$. As far as the isotropy goes, there are four points in T^2 with non-trivial stabilizer: the images of points $(0, 0)$ and $(1/2, 1/2)$ both have stabilizer \mathbb{Z}_4 , and the images of the points $(0, 1/2)$ and $(1/2, 0)$, both have stabilizer \mathbb{Z}_2 (and lie in the same σ -orbit).

We conclude that a fundamental domain for the G -action on \mathbb{R}^3 is given by the rectangular prism $P := [0, 1/2] \times [0, 1/2] \times [-1/2, 1/2] \subset \mathbb{R}^3$ (Figure 1). The interior of P gives the single 3-cell orbit for the equivariant polyhedral G -CW-structure on \mathbb{R}^3 . For the isotropy groups, we just need to understand the action on the four vertical lines lying above each of the four points $(0, 0)$, $(1/2, 0)$, $(0, 1/2)$, and $(1/2, 1/2)$. It is easy to see that the vertical line $(0, 0, z)$ consists entirely of points with stabilizer \mathbb{Z}_4 , while the vertical lines $(1/2, 0, z)$ and $(0, 1/2, z)$ both have

stabilizer \mathbb{Z}_2 . On the other hand, the action of the element of order 4 on the S^1 -fiber above the point $(1/2, 1/2)$ can be calculated, and consists of a rotation by $\pi/4$ on the S^1 -fiber. So the stabilizers for points on the line $(1/2, 1/2, z)$ are all trivial.

The last task remaining is to identify the gluings on the boundary of P . First, we have that the top and bottom squares of P are identified (via $T_z \in G$). Secondly, the two sides incident to the z -axis get “folded together” by $\sigma \in G$ (which rotates the front face $\pi/2$ radians to the left side face). Finally, the element $T_x \circ \sigma$ maps the hyperplane $y = 1/2$ (containing the back face) to the hyperplane $x = 1/2$ (containing the right side face). This element takes the line $(0, 1/2, z)$ to the line $(1/2, 0, z)$, identifying together the corresponding edges of P . On the line of intersection of these two hyperplanes, the element acts by $(1/2, 1/2, z) \mapsto (1/2, 1/2, z + 1/4)$. These give us the identifications between the faces of P , allowing us to obtain the description of $G \backslash \mathbb{R}^3$ shown in Figure 1.

EXAMPLE 17. For the group $G := \text{Hei}(\mathbb{Z}) \rtimes \mathbb{Z}_4$ described above, we have that $\text{rank}(K_0(C_r^*(G)) \otimes \mathbb{Q}) = 5$ and $\text{rank}(K_1(C_r^*(G)) \otimes \mathbb{Q}) = 5$.

Before establishing this result, we note that this is consistent with the computation by Lück, who showed that $K_n(C_r^*(G)) \cong \mathbb{Z}^5$ for all n (see [Lu3, Thm. 2.6]). This serves as a second check on our algorithm, and is, to the best of our knowledge, the only example in the literature of an explicit computation for the topological K -theory of a 3-orbifold group with non-trivial torsion.

PROOF. We apply our algorithm, using the polyhedron P described above. For the \sim equivalence classes on $F(G)$, we note that the quotient space $G \backslash \mathbb{R}^3$ has three vertices, one each with stabilizer \mathbb{Z}_4 (vertex A), \mathbb{Z}_2 (vertex B), and the trivial group (vertex C). The edges joining distinct edges all have trivial stabilizer, allowing us to identify all the identity elements together. We conclude that there are precisely five \sim equivalence classes, corresponding to the three non-trivial elements in the \mathbb{Z}_4 vertex stabilizer, the single non-trivial element in the \mathbb{Z}_2 vertex stabilizer, and the equivalence class combining all the trivial elements. This gives $\text{rank}(H_0(\mathcal{C}) \otimes \mathbb{Q}) = 5$.

Next we consider the quotient space $G \backslash \mathbb{R}^3$. The faces of P are pairwise identified, so the quotient space is a closed manifold. Moreover, with respect to the induced orientation on ∂P , the identifications between the faces are orientation reversing, so the quotient space is an orientable closed 3-manifold. Lemma 7 gives us that $H_3(\mathcal{C}) \cong \mathbb{Z}$, and hence that $\text{rank}(H_3(\mathcal{C}) \otimes \mathbb{Q}) = 1$. Note that, as mentioned earlier, [Lu3, Lemma 2.4] shows that the quotient space is actually a 3-sphere (but we do not need this fact for our computation).

The quotient space has empty boundary, so $s = t = t' = 0$. The 2-complex Y is just the 2-skeleton of the quotient space. This is the image of the boundary of P after performing the required identifications. As such, Y is constructed from two squares, a triangle, and a hexagon (see Figure 2). Note that the square corresponding to the front face of P (which also gets identified to the left face) folds up to a cylinder in Y , as its top and bottom edge get identified together (leftmost cylinder in Figure 2). The union of the hexagon and triangle, forming the back face of P (which also gets identified to the right face), similarly folds up to another cylinder in Y (rightmost cylinder in Figure 2). The two cylinders attach together along a common boundary loop (image of the edge BB) to form a single long cylinder. At one of the endpoints, the cylinder attaches to a single loop (image of the edge CC) by a degree four map. So, ignoring for the time being the last square, we

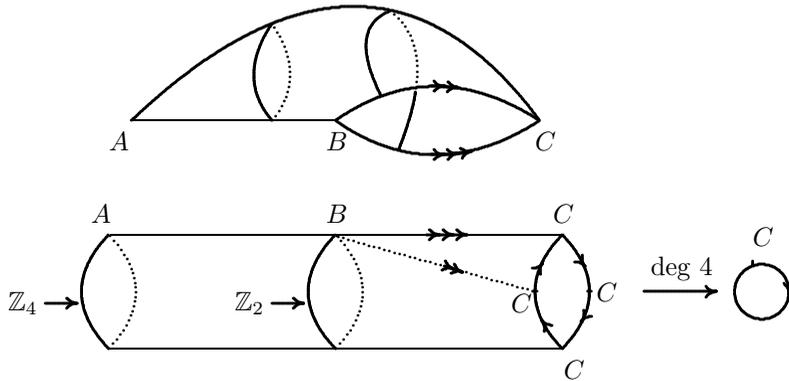


FIGURE 2. 2-skeleton of the quotient space $G \setminus \mathbb{R}^3$. The four side faces of the polyhedron P fold up into the two adjacent cylinders. On the right, the boundary circle of the cylinder gets attached to the circle by a degree 4 map. The top and bottom faces of P get identified into a single square, which attaches to the cylinder as indicated. The two loops in the cylinder based at A, B have isotropy \mathbb{Z}_4 and \mathbb{Z}_2 respectively. All remaining points have trivial isotropy.

have a subcomplex of Y which deformation retracts to S^1 (as it coincides with the mapping cylinder of the degree four map of S^1). Up to homotopy, we conclude that Y coincides with S^1 , along with a single square attached. The square comes from the top face of P (which also gets identified with the bottom face), which, after composing with the homotopy to S^1 , attaches to the S^1 via a degree one map of the boundary. This tells us that Y is homotopy equivalent to a 2-disk, and hence is contractible. By Corollary 14, we conclude that $\text{rank}(H_2(\mathcal{C}) \otimes \mathbb{Q}) = 0$.

Finally, we compute the Euler characteristic of \mathcal{C} . We have three vertices, one each with stabilizer $\mathbb{Z}_4, \mathbb{Z}_2$, and trivial. This gives an overall contribution of $+7$ to $\chi(\mathcal{C})$. We have six edges, one with stabilizer \mathbb{Z}_4 , one with stabilizer \mathbb{Z}_2 , and the remainder with trivial stabilizer. This contributes -10 to $\chi(\mathcal{C})$. There are four faces with trivial stabilizer, contributing $+4$ to $\chi(\mathcal{C})$. There is one 3-cell with trivial stabilizer, contributing -1 . Summing these up, we see that $\chi(\mathcal{C}) = 7 - 10 + 4 - 1 = 0$. From Lemma 15, we see that $\text{rank}(H_1(\mathcal{C}) \otimes \mathbb{Q}) = 4$. Applying Lemma 4, we deduce that both the rational K -groups have rank $= 5$, as claimed. \square

4.3. Hyperbolic reflection groups - I. Consider the groups $\Lambda_n, n \geq 5$, given by the following presentation:

$$\Lambda_n := \left\langle y, z, x_i, 1 \leq i \leq n \mid \begin{array}{l} y^2, z^2, \\ x_i^2, (x_i x_{i+1})^2, (x_i z)^3, (x_i y)^3, 1 \leq i \leq n \end{array} \right\rangle$$

The groups Λ_n are Coxeter groups, and the presentation given above is in fact a Coxeter presentation of the group.

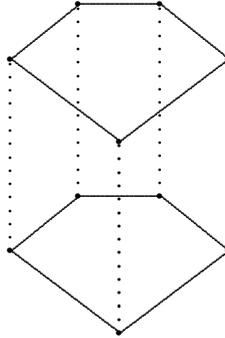


FIGURE 3. Hyperbolic polyhedron for Λ_5 . Ordinary edges have internal dihedral angle $\pi/3$. Dotted edges have internal dihedral angle $\pi/2$.

EXAMPLE 18. For the groups Λ_n whose presentations are given above,

- (1) the rank of $K_0(C_r^*(\Lambda_n)) \otimes \mathbb{Q}$ is equal to $3n + 4$,
- (2) the rank of $K_1(C_r^*(\Lambda_n)) \otimes \mathbb{Q}$ is equal to $n + 1$.

PROOF. The groups Λ_n arise as hyperbolic reflection groups, with underlying polyhedron P the product of an n -gon with an interval. This polyhedron has exactly two faces which are n -gons, and the dihedral angle along the edges of these two faces is $\pi/3$. All the remaining edges have dihedral angle $\pi/2$. An illustration of the polyhedron associated to the group Λ_5 is shown in Figure 3. We will take the Λ_n action on $X := \mathbb{H}^3$, with fundamental polyhedron P , and quotient space X/Λ_n coinciding with P . Note that this action is a model for $\underline{E}\Lambda_n$, as finite subgroups F of Λ_n have non-empty fixed sets (the center of mass of any F -orbit will be a fixed point of F), which must be convex subsets (and hence contractible). Both of these last statements are consequences of the fact that the action is by isometries on a space of non-positive curvature.

Applying the argument detailed in Section 3, we compute $\beta_0(\mathcal{C})$ by counting equivalence classes on the set $F(\Lambda_n)$. Since $X/\Lambda_n = P$, the set $F(\Lambda_n)$ consists of $2n$ copies of the group S_4 . Each individual S_4 has five conjugacy classes, given by the possible cycle structures of elements, with typical representatives: e , (12) , (123) , (1234) , $(12)(34)$. Next we consider how the edges identify the individual conjugacy classes to get the equivalence classes for \sim .

Firstly, all the individual identity elements will be identified together, yielding a single \sim class. So we will henceforth focus on non-identity classes. Each of the edges on the top n -gon has stabilizer $D_3 \cong S_3$, which has three conjugacy classes, represented by e , (12) , (123) . Under the inclusion into each adjacent vertex stabilizers, representative elements for these classes map to representative elements with the same cycle structure. So we see that all of the 3-cycles in the stabilizers

of the vertices in the top n -gon lie in the same \sim class, and likewise for all of the 2-cycles. A similar analysis applies to the vertices in the bottom n -gon. Finally, each vertical edge has stabilizer D_2 , and under the inclusion into the adjacent vertices, has image generated by the two permutations (12) and (34) (and hence identifies *three* conjugacy classes together). Putting all this together, we see that the \sim equivalence classes consist of:

- one class consisting of all the identity elements in the individual vertex groups,
- n classes of elements of order = 2, coming from the identification of cycles of the form (12)(34) for each pair of vertices joined by a vertical edge,
- one class of elements of order = 2, coming from the cycles of the form (12) in all vertex stabilizers,
- two classes of elements of order = 3, each coming from the cycles of the form (123) in the top and bottom n -gon respectively, and
- $2n$ classes of elements of order = 4, each coming from the cycles of the form (1234) in each individual vertex stabilizer.

We conclude that the $\beta_0(\mathcal{C}) = \text{rank}(H_0(\mathcal{C}) \otimes \mathbb{Q}) = 3n + 4$.

Since our quotient space $X/\Lambda_n = P$ is not a closed orientable manifold, Lemma 7 tells us that $H_3(\mathcal{C}) = 0$. To calculate $\beta_2(\mathcal{C}) = \text{rank}(H_2(\mathcal{C}) \otimes \mathbb{Q})$, we apply Corollary 14. There is a single boundary component for $X/\Lambda_n = P$, which is orientable and non-odd (it contains edges with stabilizer D_2), and contains no edges with stabilizer \mathbb{Z}_2 , so $s = 1$, $t = 0$, and $t' = 0$. Also, there are no interior 2-cells, and the single boundary component is of dihedral type, so $Y = \emptyset$. By Corollary 14, we conclude that $\text{rank}(H_2(\mathcal{C}) \otimes \mathbb{Q}) = 0$.

To calculate $\text{rank}(H_1(\mathcal{C}) \otimes \mathbb{Q})$, we need the Euler characteristic of the chain complex \mathcal{C} . There are $2n$ vertices, all with stabilizers S_4 , which each have five conjugacy classes. There are a total of $3n$ edges, n of which have stabilizer D_2 (with four conjugacy classes), and $2n$ of which have stabilizer D_3 (with three conjugacy classes). There are $n + 2$ faces, with stabilizers \mathbb{Z}_2 , which each have two conjugacy classes. There is one 3-cell, with trivial stabilizer, with a single conjugacy class. Putting this together, we have that

$$\chi(\mathcal{C}) = (5(2n)) - (3(2n) + 4(n)) + (2(n + 2)) - 1 = 2n + 3$$

Applying Lemma 15, we can now calculate:

$$\text{rank}(H_1(\mathcal{C}) \otimes \mathbb{Q}) = (3n + 4) - (2n + 3) = n + 1$$

Finally, applying Lemma 4, we obtain the desired result. □

4.4. Hyperbolic reflection groups - II. Next, let us consider a somewhat more complicated family of examples. For an integer $n \geq 2$, we consider the group Γ_n , defined by the following presentation:

$$\Gamma_n := \left\langle x_1, \dots, x_6 \mid \begin{array}{l} x_i^2, (x_1x_2)^n, (x_1x_5)^2, (x_1x_6)^2, (x_3x_4)^2, (x_2x_5)^2, (x_2x_6)^2 \\ (x_1x_4)^3, (x_2x_3)^3, (x_4x_5)^3, (x_4x_6)^3, (x_3x_5)^3, (x_3x_6)^3 \end{array} \right\rangle$$

Observe that the groups Γ_n are Coxeter groups, and that the presentation given above is in fact a Coxeter presentation of the group.

EXAMPLE 19. *For the groups Γ_n whose presentations are given above, we have that:*

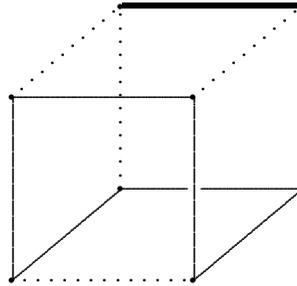


FIGURE 4. Hyperbolic polyhedron for Γ_n . Ordinary edges have internal dihedral angle $\pi/3$. Dotted edges have internal dihedral angle $\pi/2$. The thick edge has internal dihedral angle π/n .

$$\text{rank} \left(K_0(C_r^*(\Gamma_n)) \otimes \mathbb{Q} \right) = \begin{cases} \frac{3}{2}(n-1) + 12 & n \text{ odd,} \\ \frac{3}{2}n + 14 & n \text{ even,} \end{cases}$$

$$\text{rank} \left(K_1(C_r^*(\Gamma_n)) \otimes \mathbb{Q} \right) = \begin{cases} 3 & n \text{ odd,} \\ 2 & n \text{ even.} \end{cases}$$

PROOF. To verify the results stated in this example, we first observe that the Coxeter groups Γ_n arise as hyperbolic reflection groups, with underlying polyhedron P a combinatorial cube. The geodesic polyhedron associated to Γ_n is shown in Figure 4. Again, we set $X := \mathbb{H}^3$, with fundamental polyhedron P , and quotient space X/Γ_n coinciding with P . As in the previous example, X is a model for \underline{EG} .

To apply our procedure, we start by considering the equivalence relation \sim on the set $F(\Gamma_n)$. Out of the eight vertices of the cube P , six have stabilizer isomorphic to S_4 , while the remaining two have stabilizer $D_n \times \mathbb{Z}_2$. We will think of D_n as the symmetries of a regular n -gon, and let r_0, r_1 denote the reflection in a vertex, and in the midpoint of an adjacent side respectively (so r_0, r_1 are the standard Coxeter generators for D_n). Recall that the number of conjugacy classes of D_n depends on the parity of n : each rotation ϕ is only conjugate to its inverse ϕ^{-1} , while the reflections r_i fall into one or two conjugacy classes, depending on whether n is odd or even. Crossing with \mathbb{Z}_2 , each of these conjugacy class in D_n gives rise to two conjugacy classes in $D_n \times \mathbb{Z}_2$: the image class under the obvious inclusion $D_n \hookrightarrow D_n \times \mathbb{Z}_2$, and its “flipped” image, obtained by composing with the non-trivial element τ in the \mathbb{Z}_2 -factor. Next, we need to see how conjugacy classes in the individual vertex stabilizers get identified together by the edge stabilizers. After performing these identifications, we obtain that the \sim equivalence classes consist of:

- one class consisting of all the identity elements in the individual vertex groups,

- six classes of elements of order = 4, each coming from the cycles of the form (1234) in the six individual S_4 vertex stabilizers,
- one class of elements of order = 3, coming from the cycles of the form (123) in the six S_4 vertex stabilizers (these classes get identified together via the edges with stabilizer D_3),
- one class of elements of order = 2, comprised from the cycles of the form (12) in the six S_4 vertex stabilizers (identified via the edges with stabilizer D_3), along with the the three elements of the form $(r_0, 1), (r_1, 1), (1, \tau)$ in the two vertices with stabilizer $D_n \times \mathbb{Z}_2$ (identified via the edges with stabilizer D_2),
- one class of elements of order = 2, consisting of the elements of cycle form (12)(34) in the two S_4 vertex stabilizers which are joined together by an edge with stabilizer D_2 (which identifies these elements together),
- two or four classes (according to the parity of n), coming from the two elements of the form (r_0, τ) or (r_1, τ) in the two vertices with stabilizer $D_n \times \mathbb{Z}_2$ (these two elements lie in the same conjugacy class when n odd), which are each identified to elements with cycle form (12)(34) in one of the two adjacent S_4 vertex stabilizers,
- $n - 1$ or n conjugacy classes (according to n odd or even respectively), coming from elements of the form (ϕ_i, τ) in each of the two vertices with stabilizer $D_n \times \mathbb{Z}_2$, and
- $(n - 1)/2$ or $n/2$ conjugacy classes (according to n odd or even respectively), coming from the elements of the form $(\phi_i, 1)$ in the two vertices with stabilizer $D_n \times \mathbb{Z}_2$ (the elements in the two copies get identified together via the edge with stabilizer D_n).

Summing this up, we find that $\text{rank}(H_0(\mathcal{C}) \otimes \mathbb{Q})$ is $\frac{3}{2}(n - 1) + 12$ if n is odd, and $\frac{3}{2}n + 14$ if n is even.

The quotient space $X/\Gamma_n = P$ is a 3-manifold with non-empty boundary, so Lemma 7 gives us that $H_3(\mathcal{C}) = 0$. The only boundary component is orientable and non-odd, and contains no edges with stabilizer \mathbb{Z}_2 , so $s = 1$ and $t = t' = 0$. Moreover, there are no interior faces, so $Y = \emptyset$. By Corollary 14, we conclude that $\text{rank}(H_2(\mathcal{C}) \otimes \mathbb{Q}) = 0$.

Next, let us calculate the rank of $H_1(\mathcal{C}) \otimes \mathbb{Q}$. To do this, we first compute the Euler characteristic $\chi(\mathcal{C})$. We have six vertices, four with stabilizer S_4 (having five conjugacy classes), and two with stabilizer $D_n \times \mathbb{Z}_2$ (having either $n + 3$ or $n + 6$ conjugacy classes, depending on whether n is odd or even). There are twelve edges, six with stabilizer D_3 (with three conjugacy classes), five with stabilizer D_2 (with four conjugacy classes), and one with stabilizer D_n (with $(n + 3)/2$ or $(n + 6)/2$ conjugacy classes, depending on whether n is odd or even). There are six faces, each with stabilizer \mathbb{Z}_2 (with two conjugacy classes each). Finally, there is one 3-cell with trivial stabilizer. Taking the alternating sum, we obtain that the Euler characteristic is

$$\chi(\mathcal{C}) = \begin{cases} \frac{3}{2}(n - 1) + 9 & n \text{ odd,} \\ \frac{3}{2}n + 12 & n \text{ even.} \end{cases}$$

From Lemma 15, the difference between $\chi(\mathcal{C})$ and the rank of $H_0(\mathcal{C}) \otimes \mathbb{Q}$ yields the rank of $H_1(\mathcal{C}) \otimes \mathbb{Q}$, giving us that the latter is either 3 or 2 according to whether n is odd or even. Applying Lemma 4, we obtain the desired result. \square

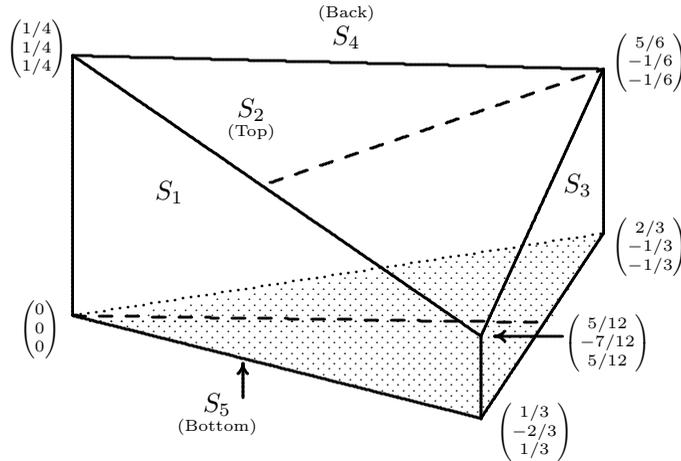


FIGURE 5. The polyhedron pictured here is an exact convex compact fundamental polyhedron for the action of G on \mathbb{R}^3 . The dashed lines represent axes of rotation (through 180 degrees) for certain elements of G . Note that the base of the figure is an equilateral triangle, but the top is only isosceles.

4.5. Crystallographic group. Our next example is taken from the work of Farley and Ortiz [FO]. Consider the lattice $L \subset \mathbb{R}^3$ generated by the three vectors

$$\mathbf{v}_1 = \begin{pmatrix} 2/3 \\ -1/3 \\ 2/3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix},$$

and let $G = \text{Sym}(L)$ denote the subgroup of $\text{Isom}(\mathbb{R}^3)$ which maps L to itself. The group G is one of the seven maximal split 3-dimensional crystallographic groups, and is discussed at length in [FO, Section 6.8].

A polyhedral fundamental domain P for the G -action on \mathbb{R}^3 is provided in Figure 5. Next we describe the stabilizers of the various faces, edges, and vertices of P (given in terms of the labeling in Figure 5).

Face stabilizers: The two triangles at the top (collectively labelled by S_2), and the two triangles at the bottom (labelled by S_5) have trivial stabilizer. The three quadrilateral sides (S_1, S_3 , and S_4) each have stabilizer \mathbb{Z}_2 , generated by the reflection in the 2-plane extending the corresponding side.

Edge stabilizers: The three vertical edges in Figure 5 each have stabilizer D_3 , generated by the reflections in the two incident faces. The two dotted edges (in the middle of the faces S_2 and S_5) have stabilizer \mathbb{Z}_2 , generated by a rotation by π centered on the edge. All remaining edges have stabilizer \mathbb{Z}_2 , generated by the reflection in the (unique) incident face whose isotropy is non-trivial. Note that, when one passes to the quotient space X/G , the two triangles in the top face S_2 get identified together by the π -rotation in the dotted line (and similarly for the two triangles in the bottom face S_5).

Vertex stabilizers: The two vertices $(0, 0, 0)$ and $(5/6, -1/6, -1/6)$ have stabilizer $D_3 \times \mathbb{Z}_2$. The two vertices $(1/4, 1/4, 1/4)$ and $(2/3, -1/3, -1/3)$ have stabilizer D_3 . Finally, the two vertices $(1/2, 1/2, 0)$ and $(1/3, -1/6, 1/3)$, the midpoints of

the edges at which the dotted lines terminate, have stabilizer D_2 . The remaining vertices of P are in the same orbit as one of the six described above.

EXAMPLE 20. *For the split crystallographic group G described above, we have that $\text{rank}(K_0(C_r^*(\Gamma_n)) \otimes \mathbb{Q}) = 12$ and $\text{rank}(K_1(C_r^*(\Gamma_n)) \otimes \mathbb{Q}) = 0$.*

PROOF. We apply our algorithm, using the polyhedron P above. Our first step is to consider the \sim equivalence relation on the set $F(G)$. The vertex and edge stabilizers for P have been described above, and the \sim equivalence classes are given as follows:

- one class consisting of all the identity elements in the individual vertex groups,
- one class consisting of all the elements of order 3 in the individual vertex groups (these occur in the four vertices with stabilizer D_3 or $D_3 \times \mathbb{Z}_2$, and are identified together via three consecutive edges with stabilizer D_3),
- one class of elements of order 2, consisting of elements of order two in the vertex groups isomorphic to D_3 , along with elements of order two in the canonical D_3 -subgroup within the vertex groups isomorphic to $D_3 \times \mathbb{Z}_2$ (these are identified together via the three consecutive edges with stabilizer D_3), and the elements of the form $(1, 0)$ in the two vertex groups isomorphic to $D_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ (identified together via the edges $S_1 \cap S_2$ and $S_3 \cap S_5$),
- two classes of elements of order 2, coming from each of the two dotted edges: the rotation by π in the edge identifies the element $(0, 1)$ in one endpoint (vertex with stabilizer $D_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$) with the element which is a product of a reflection in D_3 with a reflection in \mathbb{Z}_2 in the other endpoint (vertex with stabilizer $D_3 \times \mathbb{Z}_2$),
- six remaining classes, two each in the vertices with stabilizer $D_3 \times \mathbb{Z}_2$ and one each in those with stabilizer D_2 (these classes aren't identified to any others via the edges).

Summing this up, we see that $\text{rank}(H_0(\mathcal{C}) \otimes \mathbb{Q}) = 11$.

Next, we note that the quotient space X/G is obtained from the polyhedron P by “folding up” the top and bottom triangle along the dotted lines, resulting in \mathbb{D}^3 , a 3-manifold with non-empty boundary. Lemma 7 gives us that $H_3(\mathcal{C}) = 0$. The only boundary component is orientable and odd, and contains edges with stabilizer \mathbb{Z}_2 , so $s = t = 0$ and $t' = 1$. The 2-complex Y clearly deformation retracts to the boundary S^2 , so $\beta_2(Y) = 1$. By Corollary 14, we conclude that $\text{rank}(H_2(\mathcal{C}) \otimes \mathbb{Q}) = 1$.

Next, we calculate the rank of $H_1(\mathcal{C}) \otimes \mathbb{Q}$. As usual, we first calculate the Euler characteristic $\chi(\mathcal{C})$. We have six vertices, two with stabilizer D_2 (having four conjugacy classes), two with stabilizer D_3 (having three conjugacy classes), and two with stabilizer $D_3 \times \mathbb{Z}_2$ (having six conjugacy classes), giving an overall contribution of $+26$. There are nine edges, six with stabilizer \mathbb{Z}_2 (with two conjugacy classes), and three with stabilizer D_3 (with three conjugacy classes), giving a contribution of -21 . There are five faces, three with stabilizer \mathbb{Z}_2 (with two conjugacy classes each), and two with trivial stabilizer (with one conjugacy class each), giving a contribution of $+8$. There is one 3-cell with trivial stabilizer, contributing a -1 . Summing up these contributions, we obtain that the Euler characteristic is $\chi(\mathcal{C}) =$

$26 - 21 + 8 - 1 = 12$. From Lemma 15, we see that the rank of $H_1(\mathcal{C}) \otimes \mathbb{Q}$ is $= 0$. Applying Lemma 4, we obtain the desired result. \square

5. Concluding remarks

The examples in the previous section were chosen to illustrate our algorithm on several different types of smooth 3-orbifold groups. As the reader can see, our algorithm is quite easy to apply, once one has a good description of the orbit space $G \backslash X$. There are several natural directions for further work.

For instance, in Section 4.5, we applied our algorithm to a specific 3-dimensional crystallographic group. It is known that, in dimension $= 3$, there are precisely 219 crystallographic groups up to isomorphism. One could in principle apply our algorithm to produce a complete table of the rational K -theory groups of all 219 groups. The essential difficulty in doing this lies in finding some convenient, systematic way to identify suitable polyhedral fundamental domains for each of these groups. For the 73 *split* crystallographic groups, such fundamental domains can be found in the forthcoming paper of Farley and Ortiz [FO].

Another reasonable direction would be to focus on uniform arithmetic lattices Γ in the Lie group $PSL_2(\mathbb{C}) \cong \text{Isom}^+(\mathbb{H}^3)$. One could try to analyze the relationship (if any) between the rational K -theory of such a Γ and the underlying arithmetic structure. Again, the difficulty here lies in finding a good description of the polyhedral fundamental domain for the action (in terms of the arithmetic data).

In a different direction, one can consider *hyperbolic reflection groups*. These are groups generated by reflections in the boundary faces of a geodesic polyhedron $P \subset \mathbb{H}^3$. In this context, the polyhedron P serves as a polyhedral fundamental domain for the action, so one can readily apply our algorithm to compute the rational K -theory of the corresponding group (see the examples in Sections 4.3 and 4.4). One could try, in this special case, to refine our algorithm to produce expressions for the *integral* K -theory groups, in terms of the combinatorial data of the polyhedron P . This is the subject of an ongoing collaboration of the authors.

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Appendix A

In this Appendix, we provide the details for the computations used in some of the proofs in Section 3.4. Let $n \geq 2$ be an integer and D_n be the dihedral group with presentation

$$D_n = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^n \rangle.$$

We will compute the map

$$(3) \quad \varphi: R_{\mathbb{C}}(\mathbb{Z}_2) \oplus R_{\mathbb{C}}(\mathbb{Z}_2) \longrightarrow R_{\mathbb{C}}(D_n)$$

given by induction between representation rings with respect to the subgroups $\langle s_1 \rangle$ and $\langle s_2 \rangle$ of D_n , both isomorphic to \mathbb{Z}_2 , and opposite orientations. That is, $\varphi(\rho, \tau) = (\rho \uparrow) - (\tau \uparrow)$, where ‘ \uparrow ’ means induction between the corresponding groups.

Recall from the main text (see Section 3.4, particularly Lemma 8) that if e is a boundary edge with stabilizer D_n and σ_1 and σ_2 are incident boundary faces, then $K_0(C_r^*(G_{\sigma_i})) \cong R_{\mathbb{C}}(\mathbb{Z}_2)$ and the relevant part of the Bredon chain complex at the edge e is the map given in equation (3).

The character table for D_n is given by

D_n	$(s_1 s_2)^r$	$s_2(s_1 s_2)^r$
χ_1	1	1
χ_2	1	-1
$\widehat{\chi}_3$	$(-1)^r$	$(-1)^r$
$\widehat{\chi}_4$	$(-1)^r$	$(-1)^{r+1}$
ϕ_p	$2 \cos\left(\frac{2\pi pr}{m}\right)$	0

where $0 \leq r \leq n - 1$, p varies between 1 and $n/2 - 1$ if n is even or $(n - 1)/2$ if n is odd and the hat $\widehat{}$ denotes a character which appears only when n is even.

The character table for \mathbb{Z}_2 is given by

\mathbb{Z}_2	e	s_i
ρ_1	1	1
ρ_2	1	-1

To compute the induction homomorphism we will use Frobenius reciprocity. We first do the case $\langle s_1 \rangle$. The characters of D_n restricted to this subgroup are

	e	s_1
$\chi_1 \downarrow$	1	1
$\chi_2 \downarrow$	1	-1
$\widehat{\chi}_3 \downarrow$	1	-1
$\widehat{\chi}_4 \downarrow$	1	1
$\phi_p \downarrow$	2	0

Multiplying with the rows of the character table of $\langle s_1 \rangle \cong C_2$ we obtain the induced representations

$$\begin{aligned} \rho_1 \uparrow &= \chi_1 + \widehat{\chi}_4 + \sum \phi_p, \\ \rho_2 \uparrow &= \chi_2 + \widehat{\chi}_3 + \sum \phi_p. \end{aligned}$$

The case $\langle s_2 \rangle$ is analogous, but note that the characters 3 and 4 must be interchanged in the even case:

	e	s_j
χ_1	1	1
χ_2	1	-1
$\widehat{\chi}_3$	1	1
$\widehat{\chi}_4$	1	-1
ϕ_p	2	0

and

$$\begin{aligned} \rho_1 \uparrow &= \chi_1 + \widehat{\chi}_3 + \sum \phi_p, \\ \rho_2 \uparrow &= \chi_2 + \widehat{\chi}_4 + \sum \phi_p. \end{aligned}$$

As maps of free abelian groups we obtain

$$\begin{aligned} \mathbb{Z}^2 &\rightarrow \mathbb{Z}^{c(D_n)} \\ (a, b) &\mapsto (a, b, \widehat{b}, \widehat{a}, a + b, \dots, a + b) \quad \text{for } \langle s_1 \rangle \hookrightarrow D_n, \\ (c, d) &\mapsto (c, d, \widehat{c}, \widehat{d}, c + d, \dots, c + d) \quad \text{for } \langle s_2 \rangle \hookrightarrow D_n. \end{aligned}$$

Finally, the map φ above is

$$\begin{aligned} R_{\mathbb{C}}(\mathbb{Z}_2) \oplus R_{\mathbb{C}}(\mathbb{Z}_2) \cong \mathbb{Z}^2 \oplus \mathbb{Z}^2 &\rightarrow \mathbb{Z}^{c(D_n)} \cong R_{\mathbb{C}}(D_n) \\ (a, b, c, d) &\mapsto (a - c, b - d, \widehat{b - c}, \widehat{a - d}, S, \dots, S) \end{aligned}$$

where $S = a + b - c - d$.

As an immediate consequence of this computation, we see that if the element $\langle k, k, \dots, k \rangle$ lies in the image of ϕ , then one must have that

$$a - c = k = S = a + b - c - d.$$

Subtracting $a - c$ from both sides, we deduce that $0 = b - d = k$. In other words, the image of ϕ intersects the subgroup $\mathbb{Z} \cdot \langle 1, 1, \dots, 1 \rangle$ only in the zero vector (as was stated in Lemma 8).

Another consequence is that it is easy to identify elements in the kernel of ϕ . The equation

$$0 = (a - c, b - d, \widehat{b - c}, \widehat{a - d}, S, \dots, S)$$

forces $a = c$ and $b = d$. If in addition, n is even, then we also have $a = d$, and hence all terms must be equal. This was used in the arguments for both Lemma 10 and Lemma 12.

Appendix B

In this Appendix we compute the 2nd Betti number of the 3-manifolds M_f appearing in the Remark at the end of Section 4.1. The manifold M_f , as a mapping torus, fibers over S^1 with fiber T^2 . For this fibration, the Leray-Serre spectral sequence gives

$$E_{pq}^2 = H_p(S^1, H_q(T^2)) \Rightarrow H_{p+q}(M_f).$$

Since S^1 is 1-dimensional, $E_{p,q}^2 = 0$ unless $p = 0, 1$. The differentials have bidegree $(-2, 1)$ so the spectral sequence already collapses at the E^2 -page. This implies that

$$H_2(M_f) \cong E_{0,2}^2 \oplus E_{1,1}^2 \cong H_0(S^1, H_2(T^2)) \oplus H_1(S^1, H_1(T^2)).$$

Recall that this is not ordinary homology but rather homology with local coefficient system given by the homology of the fiber.

The homology group $H_0(S^1, H_2(T^2))$ is obtained from the chain complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\text{Id} - f_*} \mathbb{Z} \longrightarrow 0$$

where $f_*: \mathbb{Z} \rightarrow \mathbb{Z}$ is the map induced by the action of the gluing map f on the local coefficient $\mathbb{Z} = H_2(T^2)$. If $\det(\alpha) = 1$, f is orientation preserving and hence $f_* = \text{Id}$. This implies $H_0(S^1, H_2(T^2)) \cong \mathbb{Z}$. If $\det(\alpha) = -1$, f is orientation reversing and hence $f_* = -\text{Id}$. This implies $H_0(S^1, H_2(T^2)) \cong \mathbb{Z}_2$.

The homology group $H_1(S^1, H_1(T^2))$ is obtained from the chain complex

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\text{Id}-f_*} \mathbb{Z}^2 \longrightarrow 0$$

where now $f_* : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ is induced by the action of the gluing map f on the local coefficient $\mathbb{Z}^2 = H_1(T^2)$. Note that by construction f acts on $\pi_1(T^2) \cong H_1(T^2)$ via the automorphism α . So the map above is $\text{Id} - \alpha$ and hence $H_1(S^1, H_1(T^2)) \cong \ker(\text{Id} - \alpha)$. Suppose that $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\text{Id} - \alpha = \begin{pmatrix} 1-a & -b \\ -c & 1-d \end{pmatrix}$. The kernel of this map has dimension 2 if and only if $\text{Id} - \alpha = 0$, that is, $\alpha = \text{Id}$. The dimension is at least 1 if and only if the determinant is zero, that is,

$$(1-a)(1-d) = bc \Leftrightarrow 1 - \text{tr}(\alpha) + ad = bc \Leftrightarrow 1 + \det(\alpha) = \text{tr}(\alpha).$$

This occurs if and only if $\det(\alpha) = 1$ and $\text{tr}(\alpha) = 2$, or $\det(\alpha) = -1$ and $\text{tr}(\alpha) = 0$. Altogether, this gives us

$$\beta_2(M_f) = \begin{cases} 3 & \text{if } \alpha = \text{Id}, \\ 2 & \text{if } \det(\alpha) = 1, \text{tr}(\alpha) = 2, \alpha \neq \text{Id}, \\ 1 & \text{if } \det(\alpha) = 1, \text{tr}(\alpha) \neq 2, \\ 1 & \text{if } \det(\alpha) = -1, \text{tr}(\alpha) = 0, \\ 0 & \text{if } \det(\alpha) = -1, \text{tr}(\alpha) \neq 0. \end{cases}$$

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