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Relative hyperbolicity, classifying spaces, and lower algebraic K-theory

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Abstract

For Γ a relatively hyperbolic group, we construct a model for the universal space among Γ-spaces with isotropy on the family $\mathcal{VC}$ of virtually cyclic subgroups of Γ. We provide a recipe for identifying the maximal infinite virtually cyclic subgroups of Coxeter groups which are lattices in $O^+(n, 1) = \mathrm{Isom}(\mathbb{H}^n)$. We use the information we obtain to explicitly compute the lower algebraic K-theory of the Coxeter group Γ_3 (a non-uniform lattice in $O^+(3, 1)$). Part of this computation involves calculating certain Waldhausen Nil-groups for $\mathbb{Z}[D_2]$, $\mathbb{Z}[D_3]$.

Keywords: Relative hyperbolicity; Classifying spaces; Lower algebraic K-theory; Coxeter group; Hyperbolic manifold; Farrell–Jones isomorphism conjecture; Relative assembly map; Waldhausen Nil-groups; Farrell Nil-groups; Bass Nil-groups

1. Introduction

Let Γ be a discrete group and let $\mathcal{F}$ be a family of subgroups Γ. A Γ-CW-complex E is a model for the classifying space $E_{\mathcal{F}}(\Gamma)$ with isotropy on $\mathcal{F}$ if the H-fixed point sets E^H are contractible for all $H \in \mathcal{F}$ and empty otherwise. It is characterized by the universal property that for every Γ-CW-complex X whose isotropy groups are all in $\mathcal{F}$, one can find an equivariant continuous map $X \to E_{\mathcal{F}}(\Gamma)$ which is unique up to equivariant homotopy. The two extreme cases are $\mathcal{F} = \mathcal{ALL}$ (consisting of all subgroups), where $E_{\mathcal{F}}(\Gamma)$ can be taken to be a point, and $\mathcal{F} = \mathcal{TR}$ (consisting of just the trivial subgroup), where $E_{\mathcal{F}}(\Gamma)$ is a model for $E\Gamma$.

For the family of finite subgroups, the space $E_{\mathcal{F}}(\Gamma)$ has nice geometric models for various classes of groups Γ. For instance, in the case where Γ is a discrete subgroup of a virtually connected Lie group [17],

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where Γ is word hyperbolic group [29], an arithmetic group [6,40], the outer automorphism group of a free group [11], a mapping class groups [22], or a one relator group [25]. For a thorough survey on classifying spaces, we refer the reader to Lück [26].

One motivation for the study of these classifying spaces comes from the fact that they appear in the Farrell–Jones Isomorphism Conjecture about the algebraic K-theory of group rings (see [17]). Because of this conjecture the computations of the relevant K-groups can be reduced to the computation of certain equivariant homology groups applied to these classifying spaces for the family of finite groups $\mathcal{FIN}$ and the family of virtually cyclic subgroups $\mathcal{VC}$ (where a group is called *virtually cyclic* if it has a cyclic subgroup of finite index). In this paper we are interested in classifying spaces with isotropy in the family $\mathcal{VC}$ of virtually cyclic subgroups.

We start out by defining the notion of an adapted collection of subgroups associated to a pair of families $\mathcal{F} \subset \mathcal{G}$ of subgroups. We then explain how, in the presence of an adapted collection of subgroups, a classifying space $E_{\mathcal{F}}(\Gamma)$ can be modified to obtain a classifying space $E_{\mathcal{G}}(\Gamma)$ for the larger family. In the situation we are interested in, the smaller family will be $\mathcal{FIN}$ and the larger family will be $\mathcal{VC}$.

Of course, our construction is only of interest if we can find examples of groups where there is already a good model for $E_{\mathcal{FIN}}(\Gamma)$, and where an adapted family can easily be found. For Γ a relatively hyperbolic group in the sense of Bowditch [7] (or equivalently relatively hyperbolic with the bounded coset penetration property in the sense of Farb [13]), Dahmani has constructed a model for $E_{\mathcal{FIN}}(\Gamma)$. For such a group Γ, we show that the family consisting of all conjugates of peripheral subgroups, along with all maximal infinite virtually cyclic subgroups *not* conjugate into a peripheral subgroup, forms an adapted collection for the pair $(\mathcal{FIN}, \mathcal{VC})$. Both the general construction, and the specific case of relatively hyperbolic groups, are discussed in Section 2 of this paper.

In order to carry out our construction of the classifying space for the family $\mathcal{VC}$ for these groups, we need to be able to classify the maximal infinite virtually cyclic subgroups. We establish a systematic procedure to complete this classification for arbitrary Coxeter groups arising as lattices in $O^+(n, 1)$. We next focus on the group Γ_3, a Coxeter group which is known to be a non-uniform lattice in $O^+(3, 1)$. In this specific situation, it is well known that the action of Γ_3 on $\mathbb{H}^3$ is a model for $E_{\mathcal{FIN}}(\Gamma_3)$, and that the group Γ_3 is hyperbolic relative to the cusp group (in this case the 2-dimensional crystallographic group $P4m \cong [4, 4]$). Our construction now yields an 8-dimensional classifying space for $E_{\mathcal{VC}}(\Gamma_3)$. These results can be found in Section 3 of our paper.

Since the Farrell–Jones Isomorphism Conjecture is known to hold for lattices in $O^+(n, 1)$, we can use our 8-dimensional classifying space for $E_{\mathcal{VC}}(\Gamma_3)$ to compute the lower algebraic K-theory of (the integral group ring of) Γ_3. The computations are carried out in Section 4 of the paper, and yield an explicit result for $K_n(\mathbb{Z}\Gamma_3)$ when $n \le 1$. The $\tilde{K}_0(\mathbb{Z}\Gamma_3)$ and $\mathrm{Wh}(\Gamma_3)$ terms we obtain involve some Waldhausen Nil-groups.

In general, very little is known about Waldhausen Nil-groups. In Section 5, we provide a complete *explicit* determination of the Waldhausen Nil-groups that occur in $\tilde{K}_0(\mathbb{Z}\Gamma_3)$ and $\mathrm{Wh}(\Gamma_3)$. The approach we take was suggested to us by F.T. Farrell, and combined with the computations in Section 4, yields the first example of a lattice in a semi-simple Lie group for which (1) the lower algebraic K-theory is *explicitly* computed, but (2) the relative assembly map induced by the inclusion $\mathcal{FIN} \subset \mathcal{VC}$ is *not* an isomorphism. The result of our computations can be summarized in the following:

Theorem 1.1. *Let $\Gamma_3 = O^+(3, 1) \cap GL(4, \mathbb{Z})$. Then the lower algebraic K-theory of the integral group ring of Γ_3 is given as follows:*

$$\mathrm{Wh}(\Gamma_3) \cong \bigoplus_{\infty} \mathbb{Z}/2$$
$$\tilde{K}_0(\mathbb{Z}\Gamma_3) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus \bigoplus_{\infty} \mathbb{Z}/2$$
$$K_{-1}(\mathbb{Z}\Gamma_3) \cong \mathbb{Z} \oplus \mathbb{Z}, \quad \textit{and}$$
$$K_n(\mathbb{Z}\Gamma_3) \cong 0, \quad \textit{for } n < -1$$

where the expression $\bigoplus_{\infty} \mathbb{Z}/2$ *refers to a countable infinite sum of copies of* $\mathbb{Z}/2$.

This theorem corrects and completes a result of the second author [31], in which the homology $H_*^{\Gamma_3}(E_{\mathcal{FIN}}(\Gamma_3); \mathbb{KZ}^{-\infty})$ was computed. In that paper, it was incorrectly claimed that the relative assembly map

$$H_*^{\Gamma_3}(E_{\mathcal{FIN}}(\Gamma_3); \mathbb{KZ}^{-\infty}) \to H_*^{\Gamma_3}(E_{\mathcal{VC}}(\Gamma_3); \mathbb{KZ}^{-\infty})$$

was an isomorphism. We refer the reader to the erratum [32] for more details.

Finally, we note that most of the techniques developed in this paper apply in a quite general setting, and in particular to any Coxeter group that occurs as a lattice in $O^+(n, 1)$. In the present paper, we have only included the explicit computations for the group Γ_3. In a forthcoming paper [23], the authors carry out the corresponding computations for the lower algebraic K-theory of the remaining 3-simplex hyperbolic reflection groups.

2. A model for $E_{\mathcal{VC}}(\Gamma)$

Let Γ be a discrete group and $\mathcal{F}$ be a family of subgroups of Γ closed under inclusion and conjugation, i.e. if $H \in \mathcal{F}$ then $gH'g^{-1} \in \mathcal{F}$ for all $H' \subset H$ and all $g \in \Gamma$. Some examples for $\mathcal{F}$ are $\mathcal{TR}$, $\mathcal{FIN}$, $\mathcal{VC}$, and $\mathcal{ALL}$, which are the families consisting of the trivial group, finite subgroups, virtually cyclic subgroups, and all subgroups respectively.

Definition 2.1. Let Γ be any finitely generated group, and $\mathcal{F} \subset \widetilde{\mathcal{F}}$ a pair of families of subgroups of Γ, we say that a collection $\{H_\alpha\}_{\alpha \in I}$ of subgroups of Γ is *adapted* to the pair $(\mathcal{F}, \widetilde{\mathcal{F}})$ provided that:

(1) For all $G, H \in \{H_\alpha\}_{\alpha \in I}$, either $G = H$, or $G \cap H \in \mathcal{F}$.
(2) The collection $\{H_\alpha\}_{\alpha \in I}$ is *conjugacy closed* i.e. if $G \in \{H_\alpha\}_{\alpha \in I}$ then $gGg^{-1} \in \{H_\alpha\}_{\alpha \in I}$ for all $g \in \Gamma$.
(3) Every $G \in \{H_\alpha\}_{\alpha \in I}$ is *self-normalizing*, i.e. $N_\Gamma(G) = G$.
(4) For all $G \in \widetilde{\mathcal{F}} \setminus \mathcal{F}$, there exists $H \in \{H_\alpha\}_{\alpha \in I}$ such that $G \leq H$.

Remark 2.2. The collection $\{\Gamma\}$ consisting of just Γ itself is adapted to every pair $(\mathcal{F}, \widetilde{\mathcal{F}})$ of families of subgroups of Γ. Our goal in this section is to show how, starting with a model for $E_{\mathcal{F}}(\Gamma)$, and a collection $\{H_\alpha\}_{\alpha \in I}$ of subgroups adapted to the pair $(\mathcal{F}, \widetilde{\mathcal{F}})$, one can build a model for $E_{\widetilde{\mathcal{F}}}(\Gamma)$.

2.1. The construction

(1) For each subgroup H of Γ, define the induced family of subgroups $\widetilde{\mathcal{F}}_H$ of H to be $\widetilde{\mathcal{F}}_H := \{F \cap H \mid F \in \widetilde{\mathcal{F}}\}$. Note that if $g \in \Gamma$, conjugation by g maps H to $g^{-1}Hg \leq \Gamma$, and sends $\widetilde{\mathcal{F}}_H$ to $\widetilde{\mathcal{F}}_{g^{-1}Hg}$.

Let E_H be a model for the classifying space $E_{\widetilde{\mathcal{F}}_H}(H)$ of H with isotropy in $\widetilde{\mathcal{F}}_H$. Define a new space $E_{H;\Gamma} = \coprod_{\Gamma/H} E_H$. This space consists of the disjoint union of copies of E_H, with one copy for each left-coset of H in Γ. Note that E_H is contractible, but $E_{H;\Gamma}$ is not (since it is not path-connected).

(2) Next, we define a Γ-action on the space $E_{H;\Gamma}$. Observe that each component of $E_{H;\Gamma}$ has a natural H-action; we want to "promote" this action to a Γ-action. By abuse of notation, let us denote by E_{gH} the component of $E_{H;\Gamma}$ corresponding to the coset $gH \in \Gamma/H$. Fix a collection $\{g_i H | i \in I\}$ of left-coset representatives, so that we now have an identification $E_{H;\Gamma} = \coprod_{i\in I} E_{g_i H}$. Now for $g \in \Gamma$, we define the g-action on $E_{H;\Gamma}$ as follows: g maps each $E_{g_i H}$ to $E_{gg_i H} \cong E_{g_j H}$, for some $j \in I$. Recall that both $E_{g_i H}$ and $E_{g_j H}$ are copies of E_H, and that $g_j^{-1} g g_i \in H$; since H acts on E_H, we define the g-action from $E_{g_i H}$ to $E_{g_j H}$ to be:

$$\begin{tikzcd} E_{g_i H} \arrow[r, "g"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\circlearrowleft"] & E_{g_j H} \arrow[d, "\cong"] \\ E_H \arrow[r, "g_j^{-1} g g_i"'] & E_H \end{tikzcd}$$

that is the $g_j^{-1} g g_i$-action via the identification $E_{g_i H} \cong E_H \cong E_{g_j H}$. Note that while the Γ-action one gets out if this depends on the choice of left-coset representatives, different choices of representatives yield conjugate actions, i.e. there exist a Γ-equivariant homeomorphism from $\coprod_{i\in I} E_{g_i H}$ and $\coprod_{i\in I} E_{g_i' H}$. Observe that the Γ-space $E_{H;\Gamma}$ is in fact nothing more than the space obtained by applying the Borel construction $\Gamma \times_H E_H$, with the obvious left Γ-action. For conciseness, we will denote this Γ-space by $\mathrm{Ind}_H^\Gamma(E_H)$. We will be using the explicit description of $\mathrm{Ind}_H^\Gamma(E_H)$ presented here in some of our proofs.

(3) Now assume that we have a pair of families $\mathcal{F} \subseteq \widetilde{\mathcal{F}}$, and a collection $\{H_\alpha\}_{\alpha\in I}$ of subgroups of Γ adapted to the pair $(\mathcal{F}, \widetilde{\mathcal{F}})$. Select a subcollection $\{\Lambda_\alpha\}_{\alpha\in\widetilde{I}} \subseteq \{H_\alpha\}_{\alpha\in I}$ consisting of one subgroup from each conjugacy class in $\{H_\alpha\}_{\alpha\in I}$. We form the space $\widehat{X}$ by taking the join of X, a model for $E_{\mathcal{F}}(\Gamma)$, with the disjoint union of the spaces $\mathrm{Ind}_{\Lambda_\alpha}^\Gamma(E_{\Lambda_\alpha})$ defined as in (1), with the naturally induced Γ-action given in (2), i.e. $\widehat{X} = X * \coprod_{\alpha\in\widetilde{I}} \mathrm{Ind}_{\Lambda_\alpha}^\Gamma(E_{\Lambda_\alpha})$.

Theorem 2.3. *The space $\widehat{X}$ is a model for $E_{\widetilde{\mathcal{F}}}(\Gamma)$.*

Proof. We start by noting that if a space A is contractible, and B is any space, then $A * B$ is contractible; in particular the space $\widehat{X}$ is contractible (since it is a join with the contractible space X).

We need to show two points:

(1) $\widehat{X}^H$ is contractible if $H \in \widetilde{\mathcal{F}}$.
(2) $\widehat{X}^H = \emptyset$ if $H \notin \widetilde{\mathcal{F}}$.

Let us concentrate on the first point; assume $H \in \widetilde{\mathcal{F}}$. Note that if $H \in \mathcal{F}$, then since X is a model for $E_{\mathcal{F}}(\Gamma)$, we have that X^H is contractible, and since $\widehat{X}^H = (X * \coprod_{\alpha\in\widetilde{I}} \mathrm{Ind}_{\Lambda_\alpha}^\Gamma(E_{\Lambda_\alpha}))^H = X^H * \coprod_{\alpha\in\widetilde{I}} (\mathrm{Ind}_{\Lambda_\alpha}^\Gamma(E_{\Lambda_\alpha}))^H$, we conclude that $\widehat{X}^H$ is contractible.

Now assume $H \in \widetilde{\mathcal{F}} \setminus \mathcal{F}$. From property (4) of an adapted family for the pair $(\widetilde{\mathcal{F}}, \mathcal{F})$ (see Definition 2.1), there exists Λ_α for some $\alpha \in \widetilde{I}$, such that H can be conjugated into Λ_α. We now make the following claim:

Claim 2.4. $(\mathrm{Ind}_{\Lambda_\alpha}^{\Gamma}(E_{\Lambda_\alpha}))^H$ *is contractible, and* $(\mathrm{Ind}_{\Lambda_\beta}^{\Gamma}(E_{\Lambda_\beta}))^H = \emptyset$*, for all* $\beta \neq \alpha$.

Note that if we assumed Claim 2.4, we immediately get that $\widehat{X}^H$ is contractible. Indeed, since X is a model for $E_{\mathcal{F}}(\Gamma)$ and $H \notin \mathcal{F}$, $X^H = \emptyset$, and we obtain:

$$\begin{aligned}\widehat{X}^H &= (\mathrm{Ind}_{\Lambda_\alpha}^{\Gamma}(E_{\Lambda_\alpha}))^H * X^H * \coprod_{\beta\neq\alpha}(\mathrm{Ind}_{\Lambda_\beta}^{\Gamma}(E_{\Lambda_\beta}))^H \\ &= (\mathrm{Ind}_{\Lambda_\alpha}^{\Gamma}(E_{\Lambda_\alpha}))^H * \emptyset * \coprod_{\beta\neq\alpha}\emptyset \cong (\mathrm{Ind}_{\Lambda_\alpha}^{\Gamma}(E_{\Lambda_\alpha}))^H.\end{aligned}$$

Proof of Claim 2.4. We first show that, in $\mathrm{Ind}_{\Lambda_\alpha}^{\Gamma}(E_{\Lambda_\alpha}) = \coprod_{i\in I} E_{g_i\Lambda_\alpha}$, there exists a connected component that is fixed by H. Since H is conjugate to Λ_α by some $k \in \Gamma$, consider the Λ_α coset $g_i\Lambda_\alpha$ containing k. We claim that for all $h \in H$, we have $h \cdot g_i\Lambda_\alpha = g_i\Lambda_\alpha$. First observe the following easy criterion for an element $h \in \Gamma$ to map a coset $g_i\Lambda_\alpha$ to itself:

$$h \cdot g_i\Lambda_\alpha = g_i\Lambda_\alpha \iff g_i^{-1}hg_i \cdot \Lambda_\alpha = \Lambda_\alpha \iff g_i^{-1}hg_i \in \Lambda_\alpha.$$

Now in our specific situation, by the choice of $g_i\Lambda_\alpha$, we have $k = g_i\widetilde{k} \in g_i\Lambda_\alpha$, where k conjugates h into Λ_α, and $\widetilde{k} \in \Lambda_\alpha$. Since $g_i = k\widetilde{k}^{-1}$, substituting we see that $g_i^{-1}hg_i = \widetilde{k}(k^{-1}hk)\widetilde{k}^{-1}$. But by construction, we have that $k^{-1}hk \in \Lambda_\alpha$, and since $\widetilde{k} \in \Lambda_\alpha$, we conclude that $g_i^{-1}hg_i \in \Lambda_\alpha$. So by the criterion above, we see that indeed, every element of H maps the component $E_{g_i\Lambda_\alpha}$ to itself.

Our next step is to note that an element $g \in H$ with $g \neq 1$ can map *at most one* of the $E_{g_i\Lambda_\alpha}$ to itself. In fact, if $g \in H$ maps $E_{g_i\Lambda_\alpha}$ to itself and $E_{g_j\Lambda_\alpha}$ to itself, then from the definition of the action given in step (2) of the construction, we have that $H \subset g_i\Lambda_\alpha g_i^{-1}$, and $H \subset g_j\Lambda_\alpha g_j^{-1}$. From condition (1) in Definition 2.1, we have that either $g_i\Lambda_\alpha g_i^{-1} = g_j\Lambda_\alpha g_j^{-1}$, or $H \subset g_i\Lambda_\alpha g_i^{-1} \cap g_j\Lambda_\alpha g_j^{-1} \in \mathcal{F}$. But this second possibility can not occur, as $H \in \widetilde{\mathcal{F}} \setminus \mathcal{F}$. Hence we have that $g_i\Lambda_\alpha g_i^{-1} = g_j\Lambda_\alpha g_j^{-1}$. Therefore $(g_j^{-1}g_i)\Lambda_\alpha(g_i^{-1}g_j) = \Lambda_\alpha$, that is, $g_j^{-1}g_i \in N_\Gamma(\Lambda_\alpha)$. But the self-normalizing condition (see condition (3) in Definition 2.1) forces $g_j^{-1}g_i \in \Lambda_\alpha$, thus $g_i \in g_j\Lambda_\alpha$, i.e. the cosets $g_i\Lambda_\alpha$, and $g_j\Lambda_\alpha$ had to be the same to begin with.

Up to this point, we know that the H-action on $\mathrm{Ind}_{\Lambda_\alpha}^{\Gamma}(E_{\Lambda_\alpha})$ maps the $E_{g_i\Lambda_\alpha}$ component to itself, and permutes all the others. This implies $(\mathrm{Ind}_{\Lambda_\alpha}^{\Gamma}(E_{\Lambda_\alpha}))^H = E_{g_i\Lambda_\alpha}^H$. But the H-action on $E_{g_i\Lambda_\alpha}$ coincides with the $g_i^{-1}Hg_i$-action on E_{Λ_α} (see step (2) of the construction). From step (1) of the construction, since $g_i^{-1}Hg_i \in \widetilde{\mathcal{F}}_{\Lambda_\alpha}$, and E_{Λ_α} is a model for $E_{\widetilde{\mathcal{F}}_{\Lambda_\alpha}}(\Lambda_\alpha)$, we have that $E_{g_i\Lambda_\alpha}^H = E_{\Lambda_\alpha;\Gamma}^H$ is contractible. □

This completes the verification of the first point, i.e. $\widehat{X}^H$ is contractible if $H \in \widetilde{\mathcal{F}}$.

We now verify the second point, i.e. $\widehat{X}^H = \emptyset$ if $H \notin \widetilde{\mathcal{F}}$. But this is considerably easier. In fact, if $H \notin \widetilde{\mathcal{F}}$, then $H \notin \mathcal{F}$, and $X^H = \emptyset$; so let us focus on the H-action on the individual $E_{\Lambda_\alpha;\Gamma}$. By the discussion on the proof of Claim 2.4, the H-action on $E_{\Lambda_\alpha;\Gamma}$ will have empty fixed point set provided that H can not be conjugated into Λ_α. From the separability condition (1) and conjugacy closure condition (2) in the definition of an adapted family (see Definition 2.1), H can be conjugated into at most one of the Λ_α; let k^{-1} be the conjugating element, i.e. $k^{-1}Hk \subseteq \Lambda_\alpha$. Then H fixes precisely one component $E_{\Lambda_\alpha;\Gamma}$, namely the component corresponding to $E_{k\Lambda_\alpha}$. Furthermore, the H-action on $E_{k\Lambda_\alpha} \cong E_{\Lambda_\alpha}$ is via the $k^{-1}hk$-action on E_{Λ_α} which is a model for $E_{\widetilde{\mathcal{F}}_{\Lambda_\alpha}}(\Lambda_\alpha)$. Since $H \notin \widetilde{\mathcal{F}}$, then $k^{-1}Hk \notin \widetilde{\mathcal{F}}$, and this implies $k^{-1}Hk \notin \widetilde{\mathcal{F}}_{\Lambda_\alpha}$, therefore $E^H_{k\Lambda_\alpha} = \emptyset$. This immediately gives $E^H_{\Lambda_\alpha;\Gamma} = \emptyset$, and hence $\widehat{X}^H = \emptyset$. This completes the verification of the second point, and hence the proof of Theorem 2.3. □

Remark 2.5. As we mentioned in Remark 2.2, the collection $\{\Gamma\}$ consisting of just Γ itself is adapted to every pair $(\mathcal{F}, \widetilde{\mathcal{F}})$ of families of subgroups. Looking at our construction of $\widehat{X}$, and applying it to $\{\Gamma\}$, we get $\widehat{X}$ is the join of $E_{\mathcal{F}}(\Gamma)$ and $E_{\widetilde{\mathcal{F}}}(\Gamma)$. While this is indeed an $E_{\widetilde{\mathcal{F}}}(\Gamma)$, the construction is not useful, since it needs an $E_{\widetilde{\mathcal{F}}}(\Gamma)$ to produce an $E_{\widetilde{\mathcal{F}}}(\Gamma)$. In order to be useful, the construction requires a "non-trivial" adapted family of subgroups.

2.2. Relatively hyperbolic groups

Our next goal is to exhibit an adapted family of subgroups in the special case where $\mathcal{F} = \mathcal{FIN}$, $\widetilde{\mathcal{F}} = \mathcal{VC}$, and Γ is a relatively hyperbolic group (in the sense of Bowditch [7]). We refer the reader to Bowditch for the definition, and will content ourselves with mentioning that the following classes of groups are relatively hyperbolic:

(1) free products of finitely many groups (relative to the factors),
(2) geometrically finite isometry groups of Hadamard manifolds of pinched negative curvature (relative to maximal parabolic subgroups),
(3) CAT(0)-groups with isolated flats (relative to the flat stabilizers), by recent work of Hruska and Kleiner [20],
(4) fundamental groups of spaces obtained via strict relative hyperbolizations (relative to the fundamental groups of the subspaces the hyperbolization is taken relative to), by a recent paper of Belegradek [4].

Theorem 2.6. *Let Γ be a group which is hyperbolic relative to subgroups $\{H_i\}_{i=1}^k$, in the sense of Bowditch* [7] *(or equivalently, a relatively hyperbolic group with the bounded coset penetration property in the sense of Farb* [13]*). Consider the collection of subgroups of Γ consisting of:*

(1) *All conjugates of H_i (these will be called* peripheral subgroups*).*
(2) *All maximal virtually infinite cyclic subgroups V such that $V \not\subseteq gH_ig^{-1}$, for all $i = 1, \ldots k$, and for all $g \in \Gamma$.*

Then this is an adapted family for the pair $(\mathcal{FIN}, \mathcal{VC})$.

Proof. We first observe that our collection of subgroups clearly satisfies condition (4) for an adapted family (i.e. every virtually cyclic subgroup is contained in one of our subgroups). Furthermore, since the collection is conjugacy closed by construction, we see that this collection of subgroups satisfies condition (2) for an adapted family. We are left with establishing conditions (1) and (3).

Let us focus on condition (1): any two subgroups in our collection have finite intersection. A consequence of relative hyperbolicity is that any two peripheral subgroups intersect in a finite group (see Section 4 in Bowditch [7]). Hence to establish (1), it is sufficient to show:

- if V_1, V_2 are a pair of distinct maximal infinite virtually cyclic subgroups which *do not* lie inside peripheral subgroups, then $V_1 \cap V_2$ is finite, and
- if V is a maximal infinite virtually cyclic subgroup which *does not* lie in a peripheral subgroup, then it intersects each H_i in a finite subgroup.

To see the first of these two cases, assume that V_1, V_2 are as above, and that $|V_1 \cap V_2| = \infty$. Choose an element $g \in V_1 \cap V_2$ of infinite order, and note that g is *hyperbolic*, in the sense that it *cannot* be conjugated into one of the cusps. Osin has established (Theorem 4.3 in [33]) that the $\mathbb{Z}$-subgroup $\langle g \rangle$ generated by such a g lies in a *unique* maximal infinite virtually cyclic subgroup. This immediately forces $V_1 = V_2$.

Now consider the second of these two cases, and assume $V \cap H_i$ is infinite. Then picking an element $g \in V \cap H_i$ of infinite order, we observe that g is of parabolic type (since it lies in H_i). On the other hand, V contains a finite index subgroup isomorphic to $\mathbb{Z}$ generated by an element h of hyperbolic type. Now consider the intersection $\langle h \rangle \cap \langle g \rangle$, and observe that this intersection is non-empty (since both $\langle h \rangle$ and $\langle g \rangle$ are finite index subgroups in V), hence contains an element which is simultaneously of hyperbolic type and of parabolic type. But this is impossible, giving us a contradiction.

This leaves us with establishing property (3) of an adapted family. We first note that in a relatively hyperbolic group, the peripheral subgroups are self-normalizing (see Section 4 in Bowditch [7]). So we merely need to establish that the maximal infinite virtually cyclic subgroups V of hyperbolic type are self-normalizing. But Osin has established (Theorem 1.5 and Corollary 1.7 in [34]) that if $g \in \Gamma$ is hyperbolic and has infinite order, and if V is the unique maximal virtually infinite cyclic subgroup containing g, then V has the property that $V \cap gVg^{-1}$ is *finite* for every $g \in \Gamma - V$. Since V is infinite, this immediately implies that $V = N_\Gamma(V)$. This establishes property (3), and completes the proof of the theorem. □

Remark 2.7. (1) Osin has defined a notion of relative hyperbolicity in terms of relative Dehn functions. The results we cite in the proof of Theorem 2.6 make use of his definition. However, in a previous paper, Osin has established that for finitely generated groups, his notion of relative hyperbolicity coincides with Bowditch's definition (Theorem 1.5 in [33]). As such, his results apply to the setting in which we are interested.

(2) We can always view a *hyperbolic* group Γ as a group which is hyperbolic relative to the trivial subgroup $\{1\}$. In this specific case, our construction for the classifying space $E_{\mathcal{VC}}(\Gamma)$ coincides with the classifying space constructed by Pineda and Leary in [21], and independently by Lück [26].

(3) In Theorem 2.3, note that it is important that we are using Bowditch's notion of relative hyperbolic group. If we were instead using Farb's notion *without* the bounded coset penetration property (known as *weak relative hyperbolicity*), the peripheral subgroups are no longer self-normalizing. Indeed, weak relative hyperbolicity is preserved if one replaces the peripheral subgroups by subgroups of finite index (while in contrast, relative hyperbolicity is *not*).

3. The maximal infinite virtually cyclic subgroups of Γ_3

Let Γ_3 be the subgroup of $O^+(3,1)$ that preserves the standard integer lattice $\mathbb{Z}^4 \subset \mathbb{R}^{3,1}$, that is, $\Gamma_3 = O^+(3,1) \cap GL(4,\mathbb{Z})$.

Since Γ_3 is a subgroup of the discrete group $GL(4,\mathbb{Z})$, it is also a discrete subgroup of $O^+(3,1)$. The group Γ_3 is a hyperbolic, Coxeter, non-cocompact, 3-simplex reflection group with fundamental domain its defining Coxeter 3-simplex Δ^3 (see [37, pg. 301]).

The group Γ_3 is part of a nice family of discrete subgroups of isometries of hyperbolic n-space for which the Farrell–Jones Isomorphism Conjecture in lower algebraic K-theory holds, that is $H_n^{\Gamma_3}(E_{\mathcal{VC}}(\Gamma_3); \mathbb{KZ}^{-\infty}) \cong K_n(R\Gamma_3)$ for $n < 2$ (see [31, Theorem 2.1]). One of our intentions in this paper is to use this result and our model for $E_{\mathcal{VC}}(\Gamma)$ constructed in Section 2 to explicitly compute the lower algebraic K-theory of the integral group ring $\mathbb{Z}\Gamma_3$. In order to accomplish this task, we must first classify up to isomorphism the family $\mathcal{VC}$ of all virtually cyclic subgroups of Γ_3. At this point we will like to direct the reader to [31] for more information on the relatively hyperbolic groups $\Gamma_n = O^+(n,1) \cap GL(n+1,\mathbb{Z})$, for $n = 3, \ldots, 9$.

We now proceed to classify up to conjugacy all maximal virtually infinite cyclic subgroups of Γ_3 of hyperbolic type. The infinite virtually cyclic subgroups of parabolic type (or cusp groups) are virtually infinite cyclic subgroups of the cusp group $P4m$, a 2-dimensional crystallographic group isomorphic to the Coxeter group [4, 4]. These groups have already been classified by Pearson in [35, Lemma 2.3]. For the maximal virtually infinite cyclic subgroups of hyperbolic type, our approach to the classification problem is *geometric*, as opposed to previous approaches which were algebraic in nature.

Lemma 3.1. *Let $Q \le \Gamma_3$ be a infinite virtually cyclic subgroup of Γ_3 of hyperbolic type. Then there exist a geodesic $\gamma \subset \mathbb{H}^3$ such that $Q \le \mathrm{Stab}_{\Gamma_3}(\gamma)$.*

Proof. Q is infinite virtually cyclic, hence contains an infinite cyclic subgroup H of finite index. Since Q is of hyperbolic type, H stabilizes some geodesic γ in $\mathbb{H}^3$. We want to show that $Q \cdot \gamma = \gamma$. Let $g \in Q$ satisfy $g \cdot \gamma \ne \gamma$, and let $g \cdot \gamma = \gamma'$. Note that $gHg^{-1} \le Q$ stabilizes γ'. Since H and gHg^{-1} are both of finite index in Q, their intersection is an infinite cyclic subgroup, call it K. Since γ and γ' are distinct geodesics and K acts by isometries, we get $|K| < \infty$, contradicting $K \cong \mathbb{Z}$. $\square$

Note that a subgroup of the type $\mathrm{Stab}_{\Gamma_3}(\gamma)$ is always virtually cyclic. Lemma 3.1 now reduces the problem of classifying maximal virtually infinity cyclic subgroups of Γ_3 of hyperbolic type to the more geometric question of finding stabilizers of geodesics $\gamma \subset \mathbb{H}^3$.

In order to do this, first we observe that the Γ_3-action on $\mathbb{H}^3$ induces a tessellation of $\mathbb{H}^3$ by copies of the fundamental domain Δ^3 of Γ_3. This tessellation is determined by a collection of totally geodesic copies of $\mathbb{H}^2$ lying in $\mathbb{H}^3$ (each of them made up of orbits of the faces of the Δ^3) intersecting in a family of geodesics (made up by orbits of the 1-skeleton of Δ^3). We will call such a totally geodesic $\mathbb{H}^2$ a *reflecting hyperplane*. A useful fact that will be used in our stabilizer calculations is the following: given any face F of the fundamental domain Δ^3, the subgroup of Γ_3 that pointwise stabilizes F is the Coxeter subgroup generated by the reflections in the codimension one faces containing F. In particular, this subgroup has Coxeter graph the full subgraph of the Coxeter graph of Γ_3 spanned by the vertices corresponding to the codimension one faces containing F.

Now the stabilizers of the geodesic will depend on the behavior of the geodesic; more precisely, will depend on the intersection of the geodesic with the tessellation of $\mathbb{H}^3$ by copies of the fundamental domain Δ^3. Denote by $p : \mathbb{H}^3 \to \mathbb{H}^3/\Gamma_3 \cong \Delta^3$ the canonical projection from $\mathbb{H}^3$ to the fundamental domain Δ^3. We first establish two easy lemmas.

Lemma 3.2. *If $\mathrm{Stab}_{\Gamma_3}(\gamma)$ is infinite, then $p(\gamma) \subset \Delta^3$ is periodic.*

Proof. This follows from the fact that if $\mathrm{Stab}_{\Gamma_3}(\gamma)$ is virtually infinite cyclic, then it contains an element of infinite order, which must act on γ by translations. If $g \in \Gamma_3$ is this element, then $p(x) = p(g \cdot x) \in \Delta^3$, forcing periodicity of $p(\gamma)$. $\square$

Lemma 3.3. *Let $\gamma \subset \mathbb{H}^3$ be an arbitrary geodesic, $x \in \gamma$ an arbitrary point, and $g \in \mathrm{Stab}_{\Gamma_3}(\gamma) \subset \Gamma_3$ an arbitrary element. Then we have $p(x) = p(g \cdot x)$.*

Proof. This is immediate, since any two points (for instance x, and $g \cdot x$) have the same image in Δ^3 provided they differ by an element in Γ_3 (by the definition of the fundamental domain). $\square$

Definition 3.4. For γ any geodesic in $\mathbb{H}^3$, we say that:

(1) γ is of *type* I, if γ coincides with the intersection of two of the reflecting hyperplanes.
(2) γ is of *type* II, if γ is contained in a unique reflecting hyperplane.
(3) γ is of *type* III, if γ is *not* contained in any of the reflecting hyperplanes.

Note that the type of a geodesic can easily be seen in terms of its image under the projection map p. Indeed, geodesics of *type* I are those for which $p(\gamma)$ lies in the 1-skeleton of Δ^3, those of *type* II have $p(\gamma)$ lying in $\partial\Delta^3$, but not in the 1-skeleton of Δ^3, and those of *type* III have non-trivial intersection with $\mathrm{Int}(\Delta^3)$.

We now make the following easy observation: given any geodesic $\gamma \subset \mathbb{H}^3$, invariant under the isometric action of a Coxeter group Γ on $\mathbb{H}^3$, there exists a short exact sequence:

$$0 \to \mathrm{Fix}_\Gamma(\gamma) \to \mathrm{Stab}_\Gamma(\gamma) \to \mathrm{Isom}_{\Gamma,\gamma}(\mathbb{R}) \to 0,$$

where $\mathrm{Fix}_\Gamma(\gamma) \leq \Gamma$ is the subgroup of Γ that fixes γ pointwise, and $\mathrm{Isom}_{\Gamma,\gamma}(\mathbb{R})$ is the induced action of $\mathrm{Stab}_\Gamma(\gamma)$ on γ (identified with an isometric copy of $\mathbb{R}$). Furthermore we have:

(1) the group $\mathrm{Isom}_{\Gamma,\gamma}(\mathbb{R})$, being a discrete cocompact subgroup of the isometry group of $\mathbb{R}$, has to be isomorphic to $\mathbb{Z}$ or D_∞.
(2) the group $\mathrm{Fix}_\Gamma(\gamma) \leq \Gamma$ acts trivially on the geodesic γ, hence can be identified with a finite Coxeter group acting on the unit normal bundle to a point $p \in \gamma$. In particular, $\mathrm{Fix}_\Gamma(\gamma)$ is a finite subgroup of $O(2)$.

In particular, since Γ_3 is a Coxeter group, we can use this short exact sequence to get an easy description of stabilizers of type II and type III geodesics.

Proposition 3.5. *Let γ be a geodesic of type* III*, with* $\mathrm{Stab}_{\Gamma_3}(\gamma)$ *infinite virtually cyclic, them* $\mathrm{Stab}_{\Gamma_3}(\gamma)$ *is isomorphic to either $\mathbb{Z}$ or D_∞.*

Proof. Since $p(\gamma) \cap \mathrm{Int}(F) \neq \emptyset$, we have that γ enters the interior of a fundamental domain in $\mathbb{H}^3$. If $g \in \Gamma_3$ is arbitrary, then the fundamental domain and its g-translate have disjoint interiors; this forces $\mathrm{Fix}_{\Gamma_3}(\gamma) = 0$. From the short exact sequence mentioned above, we immediately obtain $\mathrm{Stab}_{\Gamma_3}(\gamma) \cong \mathbb{Z}$ or D_∞, as desired. $\square$

Proposition 3.6. *If γ is of type* II*, and* $\mathrm{Stab}_{\Gamma_3}(\gamma)$ *is virtually infinite cyclic, then* $\mathrm{Stab}_{\Gamma_3}(\gamma)$ *is either $\mathbb{Z} \times \mathbb{Z}/2$ or $D_\infty \times \mathbb{Z}/2$.*

Proof. Let us consider the short exact sequence:

$$0 \to \mathrm{Fix}_{\Gamma_3}(\gamma) \to \mathrm{Stab}_{\Gamma_3}(\gamma) \to \mathrm{Isom}_{\Gamma_3,\gamma}(\mathbb{R}) \to 0$$

where $\mathrm{Isom}_{\Gamma_3,\gamma}(\mathbb{R})$ is either $\mathbb{Z}$ or D_∞. Now let us focus on $\mathrm{Fix}_{\Gamma_3}(\gamma)$. Note that since γ is of type II, it lies in a unique reflecting hyperplane $\mathbb{H}^2 \subset \mathbb{H}^3$. We now have that $\mathrm{Fix}_{\Gamma_3}(\gamma) \cong \mathbb{Z}/2$, given by the reflection in the $\mathbb{H}^2$ containing γ. Thus $\mathrm{Stab}_{\Gamma_3}(\gamma)$ fits into one of the short exact sequences:

(1) $0 \to \mathbb{Z}/2 \to \mathrm{Stab}_{\Gamma_3}(\gamma) \to \mathbb{Z} \to 0$
(2) $0 \to \mathbb{Z}/2 \to \mathrm{Stab}_{\Gamma_3}(\gamma) \to D_\infty \to 0.$

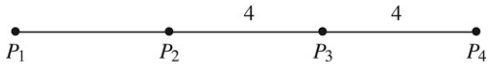

Fig. 1. The Coxeter graph of $\Gamma_3 = [3, 4, 4]$.

We next proceed to show that $\mathrm{Stab}_{\Gamma_3}(\gamma)$ is either isomorphic to $\mathbb{Z} \times \mathbb{Z}/2$ or $D_\infty \times \mathbb{Z}/2$ according to whether we are in case (1) or (2). Let us consider case (1), and pick $g \in \mathrm{Stab}_{\Gamma_3}(\gamma)$ that acts via translation along γ. Consider the reflecting hyperplane $\mathbb{H}^2 \subset \mathbb{H}^3$ containing γ, and note that $g(\mathbb{H}^2)$ is another reflecting hyperplane containing γ. But since γ is of type II there is a unique such reflecting hyperplane $\mathbb{H}^2$, hence $g(\mathbb{H}^2) = \mathbb{H}^2$, i.e. g leaves the reflecting hyperplane $\mathbb{H}^2$ containing γ invariant. This immediately implies that the subgroup generated by g commutes with the reflection in the $\mathbb{H}^2$, yielding that $\mathrm{Stab}_{\Gamma_3}(\gamma) \cong \mathbb{Z} \times \mathbb{Z}/2$, as desired. The argument in case (2) is nearly identical; pick a pair $g, h \in \mathrm{Stab}_{\Gamma_3}(\gamma)$ whose image are the generators for D_∞, and observe that g, h must be reflections in a pair of $\mathbb{H}^2$'s both of which are perpendicular to γ. This implies that the hyperplanes are both orthogonal to the reflecting hyperplane $\mathbb{H}^2$ containing γ, and hence that g, h both map the reflecting hyperplane $\mathbb{H}^2$ to itself. This yields that g, h both commute with reflection in the corresponding $\mathbb{H}^2$, and hence that $\mathrm{Stab}_{\Gamma_3}(\gamma) \cong D_\infty \times \mathbb{Z}/2$. □

To finish the classification of maximal virtually infinite cyclic subgroup of Γ_3, we are left with the study of stabilizers of geodesic of type I. Recall that geodesics γ of type I are those for which $p(\gamma)$ lies in the 1-skeleton of Δ^3. In this situation, the short exact sequence is not too useful for our purposes. Instead, we switch our viewpoint, and appeal instead to Bass–Serre theory [41].

Recall that the group Γ_3 is a Coxeter group with Coxeter graph given in Fig. 1. The fundamental domain for the Γ_3-action on $\mathbb{H}^3$ is a 3-simplex Δ^3 in $\mathbb{H}^3$, with one ideal vertex. The group $\Gamma_3 \leq \mathrm{Isom}(\mathbb{H}^3)$ is generated by a reflection in the hyperplanes (totally geodesic $\mathbb{H}^2$) extending the four faces of the 3-simplex. Each reflection corresponds to a generator, and the faces of the 3-simplex are labelled $P_1, \ldots, P_4$ according to the corresponding generator. Note that the angles between the faces can be read off from the Coxeter graph. The possible type I geodesics correspond to geodesics extending the intersections of pairs of faces (and hence, there are at most six such geodesics).

Proposition 3.7. *Let γ be the geodesic extending the intersection of the two hyperplanes $P_1 \cap P_3$, and let η be the geodesic extending $P_1 \cap P_2$. Then* $\mathrm{Stab}_{\Gamma_3}(\gamma) \cong D_2 \times D_\infty$, *and* $\mathrm{Stab}_{\Gamma_3}(\eta) \cong D_3 \times D_\infty$. *Furthermore, for the geodesics extending the remaining edges in the one skeleton, the stabilizers are finite.*

Proof. Our procedure for identifying the stabilizers of the geodesics of type I relies on the observation that when these stabilizers act cocompactly on the geodesics, Bass–Serre theory gives us an easy description of the corresponding stabilizer. Indeed, if the quotient of the geodesic in the fundamental domain is a segment, then the stabilizer of the geodesic is an amalgamation of the vertex stabilizers, amalgamated over the edge stabilizer. Let us carry out this procedure in the specific case of Γ_3. Notice that three of the six edges in the 1-skeleton of the fundamental domain Δ^3 are actually geodesic rays (since one of the vertices is an ideal vertex), and hence their geodesic extensions will have finite stabilizers. This leaves us with three edges in the 1-skeleton to worry about, namely those corresponding to the edges $P_1 \cap P_2$, $P_1 \cap P_3$, and $P_1 \cap P_4$. Let us first focus on the edge corresponding to the intersection $P_1 \cap P_4$. We claim that the geodesic extending this edge projects into a non-compact segment inside Δ^3. Indeed, if one considers the link of the vertex $P_1 \cap P_2 \cap P_4$ (a 2-dimensional sphere), we note that it has

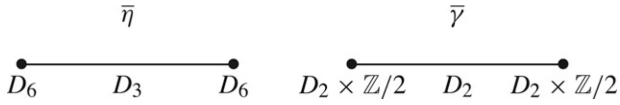

Fig. 2. Graph of groups for Type I subgroups.

a natural action by a parabolic subgroup (in the sense of Coxeter groups), namely the stabilizer of this vertex. Corresponding to this action is a tessellation of S^2 by geodesic triangles, where the vertices of the triangles correspond to the directed edges from the given vertex to (translates) of the remaining vertices from the fundamental domain. In other words, each vertex of the tessellation of S^2 comes equipped with a *label* identifying which directed edge in the fundamental domain it corresponds to. But now in the tessellation of the link of the vertex $P_1 \cap P_2 \cap P_4$, it is easy to see that the vertex antipodal to the one corresponding to the edge $P_1 \cap P_4$, has a label indicating that it projects to the edge $P_2 \cap P_4$. This immediately allows us to see that the geodesic ray extending the edge $P_1 \cap P_4$ does *not* project to a closed loop in Δ^3, and hence has finite stabilizer.

So we are left with identifying the stabilizers of the two geodesics γ and η. Let us consider the geodesic γ, and observe that the segment that is being extended joins the vertex $P_1 \cap P_2 \cap P_3$ to the vertex $P_1 \cap P_3 \cap P_4$. When looking at the tilings of S^2 one obtains at each of these two vertices, we find that in both cases, the label of the vertex corresponding to the edge $P_1 \cap P_3$ is antipodal to a vertex with the same labeling. This tells us that the geodesic γ projects precisely to the segment $P_1 \cap P_3$ in the fundamental domain Δ^3. Now the subgroup of $\mathrm{Stab}_{\Gamma_3}(\gamma)$ that fixes the vertex $P_1 \cap P_2 \cap P_3$ is the subgroup of the Coxeter group stabilizing the vertex, that additionally fixes the pair of antipodal vertices in the tiling of S^2 corresponding to γ. But it is immediate by looking at the tiling that this subgroup at each of the two vertices is just $D_2 \times \mathbb{Z}_2$, where in both cases the D_2 stabilizer of the edge $P_1 \cap P_3$ injects into the first factor, and the $\mathbb{Z}_2$-factor is generated by the reflection of S^2 that interchanges the given antipodal points. This yields that $\mathrm{Stab}_{\Gamma_3}(\gamma)$ is the amalgamation of two copies of $D_2 \times \mathbb{Z}_2$ along the D_2 factors, giving us $D_2 \times D_\infty$ (see Fig. 2 for the graph of groups). An identical analysis in the case of the geodesic η yields that the stabilizer is an amalgamation of two copies of $D_3 \times \mathbb{Z}_2$ along the D_3 factors, yielding that the stabilizer is $D_3 \times D_\infty$ (see Fig. 2). This completes the proof of Proposition 3.7. □

4. The computations of $\mathrm{Wh}_n(\Gamma_3)$, $n < 2$

In this section we briefly recall the Isomorphism Conjecture in lower algebraic K-theory (the interested reader should refer to [17,12]), and compute the homology groups $H_n^{\Gamma_3}(E_{\mathcal{VC}}(\Gamma_3); \mathbb{K}\mathbb{Z}^{-\infty}) \cong \mathrm{Wh}_n(\Gamma_3)$ for $n < 2$.

The *Farrell and Jones Isomorphism Conjecture in algebraic K-theory*, reformulated in terms of the Davis and Lück functor $\mathbb{K}R^{-\infty}$ (see [12]), states that the assembly map $H_n^{\Gamma}(E_{\mathcal{VC}}(\Gamma); \mathbb{K}R^{-\infty}) \longrightarrow K_n(R\Gamma)$ is an isomorphism for all $n \in \mathbb{Z}$.

The main point of the validity of this conjecture is that it allows the computations of the groups of interest $K_n(R\Gamma)$ from the values of $\mathbb{K}R^{-\infty}(\Gamma/H)$ on the groups $H \in \mathcal{VC}$.

The pseudo-isotopic version of the Farrell and Jones Conjecture is obtained by replacing the algebraic K-theory spectrum by the functors $\mathcal{P}_*$, $\mathcal{P}_*^{\mathrm{diff}}$, which map from the category of topological spaces X to the category of $\Omega - \mathcal{SPECTRA}$. The functor $\mathcal{P}_*(?)$ (or $\mathcal{P}_*^{\mathrm{diff}}(?)$) maps the space X to the Ω-spectrum of stable topological (or smooth) pseudo-isotopies of X (see [17, Section 1.1]).

The relation between $\mathcal{P}_*(?)$ and lower algebraic K-theory is given by the work of Anderson and Hsiang [2, Theorem 3]. They show

$$\pi_j(\mathcal{P}_*(X)) = \begin{cases} \mathrm{Wh}(\pi_1(X)), & j=-1 \\ \tilde{K}_0(\mathbb{Z}\pi_1(X)), & j=-2 \\ K_{j+2}(\mathbb{Z}\pi_1(X)), & j \le -3. \end{cases}$$

The main result in [17] is that the Isomorphism Conjecture is true for the pseudo-isotopy and smooth pseudo-isotopy functors when $\pi_1(X) = \Gamma$ is a discrete cocompact subgroup of a virtually connected Lie group. This result together with the identification given by Anderson and Hsiang of the lower homotopy groups of the pseudo-isotopy spectrum and the lower algebraic K-theory implies the following theorem (see [17, Section 1.6.5, and Theorem 2.1]):

Theorem 4.1 (*F.T. Farrell and L. Jones*). *Let Γ be a cocompact discrete subgroup of a virtually connected Lie group. Then the assembly map*

$$H_n^{\Gamma}(E_{\mathcal{VC}}(\Gamma); \mathbb{KZ}^{-\infty}) \longrightarrow K_n(\mathbb{Z}\Gamma)$$

is an isomorphism for $n \le 1$ and a surjection for $n = 2$.

Farrell and Jones also proved Theorem 4.1 for discrete cocompact groups, acting properly discontinuously by isometries on a simply connected Riemannian manifold M with everywhere non-positive curvature [17, Proposition 2.3]. Berkove, Farrell, Pineda, and Pearson extended this result to discrete groups, acting properly discontinuously on hyperbolic n-spaces via isometries, whose orbit space has finite volume (but is not necessarily compact), (see [5, Theorem A]). In particular this result is valid for Γ a hyperbolic, non-cocompact, n-simplex reflection group.

Therefore for $\Gamma = \Gamma_3$, it follows that

$$H_n^{\Gamma_3}(E_{\mathcal{VC}}(\Gamma_3); \mathbb{KZ}^{-\infty}) \cong \mathrm{Wh}_n(\Gamma_3), \quad \text{for } n < 2.$$

Hence to compute the lower algebraic K-theory of the integral group ring of Γ_3 it suffices to compute for $n < 2$, the homotopy groups

$$H_n^{\Gamma_3}(E_{\mathcal{VC}}(\Gamma_3); \mathbb{KZ}^{-\infty}).$$

These computations are feasible using the Atiyah–Hirzebruch type spectral sequence described by Quinn in [36, Theorem 8.7] for the pseudo-isotopy spectrum $\mathcal{P}$ (see Section 5):

$$E^2_{p,q} = H_p(E_{\mathcal{F}}(\Gamma)/\Gamma; \{\mathrm{Wh}_q(\Gamma_\sigma)\}) \Longrightarrow \mathrm{Wh}_{p+q}(\Gamma),$$

where

$$\mathrm{Wh}_q(F) = \begin{cases} \mathrm{Wh}(F), & q=1 \\ \tilde{K}_0(\mathbb{Z}F), & q=0 \\ K_n(\mathbb{Z}F), & q \le -1. \end{cases}$$

All the information needed to compute the E^2 term is encoded in $E_{\mathcal{VC}}(\Gamma_3)/\Gamma_3$ and the algebraic K-groups of the finite subgroups and maximal infinite virtually cyclic subgroups of Γ_3.

We now proceed to give a proof of our main theorem (Theorem 1.1). Recall that this theorem states that the lower algebraic K-theory of the integral group ring of Γ_3 is given as follows:

$$\mathrm{Wh}(\Gamma_3) \cong \bigoplus_{\infty} \mathbb{Z}/2$$
$$\tilde{K}_0(\mathbb{Z}\Gamma_3) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus \bigoplus_{\infty} \mathbb{Z}/2$$
$$K_{-1}(\mathbb{Z}\Gamma_3) \cong \mathbb{Z} \oplus \mathbb{Z}, \quad \text{and}$$
$$K_n(\mathbb{Z}\Gamma_3) \cong 0, \quad \text{for } n < -1.$$

Proof. Since Γ_3 is hyperbolic relative to $P4m$, then the fundamental domain Δ^3 has one cusp with cusp subgroup isomorphic to $P4m$, which is the unique maximal parabolic subgroup of Γ_3 (up to conjugacy). We recall that the group $P4m$ is the 2-dimensional crystallographic group generated by reflections in the sides of a Euclidean triangle with two angles $= \pi/4$. In terms of the Coxeter diagram for Γ_3 given in Fig. 1, the subgroup $P4m$ arises as the stabilizer of the (ideal) vertex corresponding to the intersection of the three faces $P_2 \cap P_3 \cap P_4$, and hence is the Coxeter group $[4, 4]$.

Let $X_0 \subset \mathbb{H}^3$ be the space obtained by truncating the cusp, that is, we remove from $\mathbb{H}^3$ a countable collection of pairwise disjoint (open) horoballs, in a Γ_3-equivariant way, where the horoballs B_i are based at the points $\{p_i\} \subset \partial^{\infty}\mathbb{H}^3 = S^2$ which are fixed points of the subgroups $G \leq \Gamma_3$ which are conjugate to $P4m$. Now let $X = X_0/\Gamma_3$, i.e. $X = \Delta^3 \setminus B_i$, where B_i is the open horoball from our countable collection that is based at the ideal vertex of Δ^3. Note that X is a 3-simplex with one vertex truncated (resulting in a triangular prism).

Note that X_0 is a Γ_3-equivariant deformation retract of the $\mathbb{H}^3$, and hence has the same equivariant homotopy type as $\mathbb{H}^3$. Since $\mathbb{H}^3$ is a model for $E_{\mathcal{FIN}}(\Gamma_3)$, we see that X_0 satisfies the requirements to be a model for $E_{\mathcal{FIN}}(\Gamma_3)$. The quotient space X has five faces (with stabilizers: $1, \mathbb{Z}/2$), nine edges (with stabilizers: $\mathbb{Z}/2$, D_2, D_3, D_4), and six vertices (with stabilizers: D_2, D_4, D_6, $\mathbb{Z}/2 \times D_4$, $\mathbb{Z}/2 \times S_4$).

Let $\mathcal{VC}_\infty$ be a collection of subgroups of Γ_3, consisting of one representative from each conjugacy class of maximal virtually cyclic subgroups of Γ_3 of hyperbolic type. Note that for each $H \in \mathcal{VC}_\infty$, a model for $E_{\mathcal{VC}}(H)$ can be taken to be a single point with the obvious H-action. Now let $E_{\mathcal{VC}}(P4m)$ be the classifying space for $P4m$ constructed by Alves and Ontaneda in [1]. $E_{\mathcal{VC}}(P4m)$ is a 4-dimensional CW-complex, and has isotropy groups (see [1]) in each dimension given by subgroups isomorphic to:

(1) *Isotropy of the* 0-*cells*:
 (a) *Finite subgroups*: $1, \mathbb{Z}/2, D_2, D_4$.
 (b) *Infinite virtually cyclic subgroups*: $\mathbb{Z}$, D_∞, $\mathbb{Z} \times \mathbb{Z}/2$, $D_\infty \times \mathbb{Z}/2$.
(2) *Isotropy of the* 1-*cells*:
 (a) *Finite subgroups*: $1, \mathbb{Z}/2, D_2$.
 (b) *Infinite virtually cyclic subgroups*: $\mathbb{Z}$, D_∞.
(3) *Isotropy of the* 2-*cells*: $1, \mathbb{Z}/2$.
(4) *Isotropy of the* 3-*cells and* 4-*cells*: trivial.

For the convenience of the reader, we include in the Appendix a brief description of the classifying space for the crystallographic group $P4m$, as constructed by Alves and Ontaneda [1]. We also briefly describe the computation of the stabilizers appearing in the above list.

Let us define the Γ_3-space $Y = (\coprod_{H \in \mathcal{VC}_\infty} \mathrm{Ind}_H^{\Gamma_3}(*)) \amalg \mathrm{Ind}_{P4m}^{\Gamma_3}(E_{\mathcal{VC}}(P4m))$. Finally, we form the Γ_3-space:

$$\widehat{X} = X_0 * Y = X_0 * \left[\left(\coprod_{H \in \mathcal{VC}_\infty} \mathrm{Ind}_H^{\Gamma_3}(*) \right) \amalg \mathrm{Ind}_{P4m}^{\Gamma_3}(E_{\mathcal{VC}}(P4m)) \right].$$

From Theorems 2.3 and 2.6, we have that $\widehat{X}$ satisfies the requirements to be a model for $E_{\mathcal{VC}}(\Gamma_3)$. Now the space $\widehat{X}$ is an 8 dimensional Γ_3-CW-complex, and has the property that any k-dimensional cell σ^k in $\widehat{X}$ is the join of an i-dimensional cell τ^i in X_0 (where $-1 \le i \le k$) with a $(k-i-1)$-dimensional cell τ^{k-i-1} in Y. Since the Γ_3-action on $\widehat{X}$ is the join of the Γ_3-actions on the two components X_0 and Y, we immediately obtain:

$$\mathrm{Stab}_{\Gamma_3}(\sigma^k) = \mathrm{Stab}_{\Gamma_3}(\tau^i) \cap \mathrm{Stab}_{\Gamma_3}(\tau^{k-i-1}).$$

This allows us to easily compute the isotropy groups of $\widehat{X}$ by taking intersections. The isotropy groups (up to isomorphism) we obtain for $\widehat{X}$ are the following:

(1) *Isotropy of the* 0-*cells*
  (a) *Finite subgroups*: $1, \mathbb{Z}/2, D_2, D_4, D_6, \mathbb{Z}/2 \times D_4, \mathbb{Z}/2 \times S_4$.
  (b) *Infinite subgroups*: $\mathbb{Z}$, D_∞, $\mathbb{Z} \times \mathbb{Z}/2$, $D_\infty \times \mathbb{Z}/2$, $D_2 \times D_\infty \cong D_2 \times \mathbb{Z}/2 *_{D_2} D_2 \times \mathbb{Z}/2$, $D_3 \times D_\infty \cong D_6 *_{D_3} D_6$.
(2) *Isotropy of the* 1-*cells*:
  (a) *Finite subgroups*: $1, \mathbb{Z}/2, D_2, D_3, D_4, D_6, D_2 \times \mathbb{Z}/2$.
  (b) *Infinite subgroups*: $\mathbb{Z}$, D_∞.
(3) *Isotropy of the* 2-*cells*: $1, \mathbb{Z}/2, D_2, D_3, D_4$.
(4) *Isotropy of the* 3-*cells*: $1, \mathbb{Z}/2, D_2$.
(5) *Isotropy of the* 4-*cells, and* 5-*cells*: $1, \mathbb{Z}/2$.
(6) *Isotropy of the* 6-*cells*, 7-*cells and* 8-*cells*: 1.

The complex that gives the homology of $E_{\mathcal{VC}}(\Gamma_3)/\Gamma_3$ with local coefficients $\{\mathrm{Wh}_q(F_\sigma)\}$ has the form

$$\bigoplus_{\sigma^8} \mathrm{Wh}_q(F_{\sigma^8}) \to \cdots \to \bigoplus_{\sigma^2} \mathrm{Wh}_q(F_{\sigma^2}) \to \bigoplus_{\sigma^1} \mathrm{Wh}_q(F_{\sigma^1}) \to \bigoplus_{\sigma^0} \mathrm{Wh}_q(F_{\sigma^0}),$$

where σ^i denotes the cells in dimension i. The homology of this complex will give us the entries for the E^2-term of the spectral sequence. The rest of this paper will focus on computing the E^2-terms of this spectral sequence.

In order to facilitate the reading of this paper, let us point out that for most of the groups F_{σ^i} occurring as stabilizers of cells, the groups $\mathrm{Wh}_q(F_{\sigma^i})$ are trivial. For the convenience of the reader, we list below the only stabilizers (and their multiplicity in the chain complex) that will have non-zero contributions to the homology of the chain complex mentioned above.

1. 0-*cell stabilizers*: $D_6, \mathbb{Z}/2 \times D_4, \mathbb{Z}/2 \times S_4, D_2 \times D_\infty$, and $D_3 \times D_\infty$, each of which occurs once.
2. 1-*cell stabilizers*: D_6 occurring twice.

Let us briefly outline how these various cell stabilizers can be seen geometrically. The two 0-cell stabilizers $D_2 \times D_\infty$, and $D_3 \times D_\infty$ arise as stabilizers of the 0-cells in the space Y corresponding to the two geodesics γ and η which are the only type I hyperbolic geodesics (see Proposition 3.7). The 0-cell stabilizers D_6, $\mathbb{Z}/2 \times D_4$ and $\mathbb{Z}/2 \times S_4$ arise as stabilizers of vertices for the Γ_3-action on the space

X_0; in terms of the fundamental domain, these vertices are the intersections of the faces $P_1 \cap P_2 \cap P_4$, $P_1 \cap P_3 \cap P_4$ and $P_1 \cap P_2 \cap P_3$, respectively (with respect to the Coxeter diagram in Fig. 1).

The two copies of D_6 as 1-cell stabilizers occur in the following manner: in the space $E_{\mathcal{FIN}}(\Gamma_3)$, there are two copies of D_6 arising as subgroups of stabilizers of 0-cells, namely the stabilizers of the endpoints w_1, w_2 of the geodesic segment $\bar{\eta}$ in Fig. 2, which *additionally* map the geodesic η to itself. Note that corresponding to the geodesic η, we have a vertex v in the Γ_3-space Y, hence when we form the join $X_0 * Y$ to obtain $E_{\mathcal{VC}}(\Gamma_3)$, we obtain a pair of 1-cells from the join of v with the two points $w_1, w_2 \in X_0$, each of which has stabilizer D_6.

In particular, as we shall see, the chain complex above reduces to one of the form:

$$0 \to \bigoplus_{\sigma^1} \mathrm{Wh}_q(F_{\sigma^1}) \to \bigoplus_{\sigma^0} \mathrm{Wh}_q(F_{\sigma^0}) \to 0.$$

More precisely, we will see that the only non-zero terms arising from the homology of this chain complex will occur for the values (p, q) equal to $(0, 1)$, $(0, 0)$, and $(0, -1)$, giving us that the only non-zero terms in the spectral sequence are the terms $E^2_{0,1}$, $E^2_{0,0}$, and $E^2_{0,-1}$. This will immediately give us that the spectral sequence collapses at the E^2-term, which in turn, will allow us to read off the lower algebraic K-theory of Γ_3 from the E^2-terms of the spectral sequence.

We now proceed to analyze the chain complex for the E^2-terms of the spectral sequence, with the goal of justifying the statements made in the previous paragraphs. We break down the analysis into four distinct cases: $q < -1$, and $q = -1, 0, 1$.

$q < -1$. Carter showed in [8] that $K_q(\mathbb{Z}F) = 0$ when F is a finite group. Farrell and Jones showed in [18] that $K_q(\mathbb{Z}Q) = 0$ when Q is a infinite virtually cyclic group. Hence the whole complex consists of zero terms and we obtain $E^2_{p,q} = 0$ for $q < -1$.

$q = -1$. Again using Carter's result in [8], $K_{-1}(\mathbb{Z}F) = 0$, for all the finite subgroups which occur as stabilizers of the n-cells, with $n = 2, \ldots, 8$. Therefore $E^2_{p,-1} = 0$ for $p \geq 2$. Also $K_{-1}(\mathbb{Z}F) = 0$, for all finite subgroups which are stabilizers of the 1-cells except for $F = D_6$; we have two 1-cells with stabilizer $F = D_6$, for which $K_{-1}(\mathbb{Z}D_6) = \mathbb{Z}$ (see [35, pg. 274]). For $n = 1$, we also have 1-cells with stabilizers isomorphic to $\mathbb{Z}$ or D_∞. But it is well known that $K_n(\mathbb{Z}Q)$ for $n < 0$ vanishes if $Q = \mathbb{Z}$ or $Q = D_\infty$ (see [3]). It follows that for $p = 0, 1$ the complex may have non-zero terms only in dimension one and zero:

$$0 \to \bigoplus_{\sigma^1} K_{-1}(D_6) \to \left[\bigoplus_{F_{\sigma^0} \in \mathcal{FIN}} K_{-1}(\mathbb{Z}F_{\sigma^0}) \oplus \bigoplus_{Q_{\sigma^0} \in \mathcal{VC}_\infty} K_{-1}(\mathbb{Z}Q_{\sigma^0}) \right].$$

If F_{σ^0} is one of the finite subgroups groups $\mathbb{Z}/2$, D_2, or D_4, then $K_{-1}(\mathbb{Z}F) = 0$ (see [8]). As we mentioned earlier, $K_{-1}(\mathbb{Z}D_6) = \mathbb{Z}$ (see [35]). If $F = \mathbb{Z}/2 \times D_4$, or $F = \mathbb{Z}/2 \times S_4$, then Ortiz in [31, pg. 350]) showed that $K_{-1}(\mathbb{Z}[\mathbb{Z}/2 \times D_4]) = 0$, and $K_{-1}(\mathbb{Z}[\mathbb{Z}/2 \times S_4]) = \mathbb{Z}$.

If $Q_{\sigma^0} \in \mathcal{VC}_\infty$, then Q_{σ^0} is one of the groups: $\mathbb{Z}$, D_∞, $\mathbb{Z} \times \mathbb{Z}/2$, $D_\infty \times \mathbb{Z}/2$, $D_2 \times D_\infty \cong D_2 \times \mathbb{Z}/2 *_{D_2} D_2 \times \mathbb{Z}/2$, $D_3 \times D_\infty \cong D_6 *_{D_3} D_6$. As we mentioned earlier, the groups $K_n(\mathbb{Z}Q)$ for $n < 0$ vanish if $Q = \mathbb{Z}$, or $Q = D_\infty$ (see [3]). For the groups with $\mathbb{Z}/2$ summands, we also have that $K_n(\mathbb{Z}Q) = 0$ if $n < 0$, (see [35, pg. 272]).

The other two cases are groups of the form $Q = Q_0 *_{D_n} Q_1$ $(n = 2, 3)$. In [18] Farrell and Jones show that if Q is infinite virtually cyclic, then $K_n(\mathbb{Z}Q)$ is zero for $n < -1$ and that $K_{-1}(\mathbb{Z}Q)$ is generated by the images of $K_{-1}(\mathbb{Z}F)$ where F ranges over all finite subgroups $F \subset Q$. Since $\mathrm{Wh}_q(D_2) = 0$ for $q < 0$, and $\mathrm{Wh}_q(D_2 \times \mathbb{Z}/2) = 0$ for $q < 0$ (see [27]), then for $Q = D_2 \times D_\infty \cong D_2 \times \mathbb{Z}/2 *_{D_2} D_2 \times \mathbb{Z}/2$, it follows that $K_{-1}(\mathbb{Z}Q) = 0$. Since $K_{-1}(\mathbb{Z}D_3) = 0$, and $K_{-1}(\mathbb{Z}D_6) = \mathbb{Z}$ (see [31,35]), then for $Q = D_3 \times D_\infty \cong D_6 *_{D_3} D_6$, it follows that $K_{-1}(\mathbb{Z}Q) = \mathbb{Z} \oplus \mathbb{Z}$.

Now note that (1) there is only one occurrence for each of the subgroups of Γ_3 occurring as stabilizers of the 0-cells (and contributing non-trivially to the chain complex), and (2) there are two occurrences of the group D_6 arising as stabilizers of the 1-cells (and the remaining stabilizers of 1-cells contribute trivially to the chain complex). Applying this to the complex that gives the homology groups $H_0(\widehat{X}; \{K_{-1}(\mathbb{Z}F_\sigma)\})$, and $H_1(\widehat{X}; \{K_{-1}(\mathbb{Z}F_\sigma)\})$ yields the following

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to 0.$$

Hence after working through the above chain complex, we have that the E^2 term for $\widehat{X}$ has the following entries for $q = -1$: $E^2_{p,q} = 0$ for $p \geq 1$, and $E^2_{0,-1} = \mathbb{Z} \oplus \mathbb{Z}$.

$q = 0$. It is well known that $\tilde{K}_0(\mathbb{Z}F) = 0$ when F is any of the finite subgroups that occur as stabilizers of the n-cells, for $n = 1, \ldots, 8$ (see for example [38,39]). For $n = 1$, we have 1-cells with stabilizers which are infinite virtually cyclic subgroups isomorphic to $\mathbb{Z}$ and D_∞. As mentioned earlier, it is well known that $\tilde{K}_0(\mathbb{Z}Q) = 0$ if $Q = \mathbb{Z}$ or D_∞ (see [3]), therefore $E^2_{p,0} = 0$ for $p \geq 1$. For $p = 0$ the complex may have non-zero terms in dimension zero and the resulting homology is:

$$H_0(\widehat{X}; \{\tilde{K}_0(\mathbb{Z}F_\sigma)\}) = \bigoplus_{F_{\sigma^0} \in \mathcal{FIN}} \tilde{K}_0(\mathbb{Z}F_{\sigma^0}) \oplus \bigoplus_{Q_{\sigma^0} \in \mathcal{VC}_\infty} \tilde{K}_0(\mathbb{Z}Q_{\sigma^0}).$$

If F_{σ^0} is one of the finite subgroups $\mathbb{Z}/2$, D_2, D_4, or D_6, then $\tilde{K}_0(\mathbb{Z}F) = 0$ (see [38]). If $F = \mathbb{Z}/2 \times D_4$, or $F = \mathbb{Z}/2 \times S_4$, Ortiz in [31, pg. 351] showed that $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/2 \times D_4]) = \mathbb{Z}/4$, and $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/2 \times S_4]) = \mathbb{Z}/4$.

If Q_{σ^0} is one of the maximal infinite virtually cyclic subgroups $\mathbb{Z}$, D_∞, $\mathbb{Z} \times \mathbb{Z}/2$, or $D_\infty \times \mathbb{Z}/2$, then $\tilde{K}_0(\mathbb{Z}Q) = 0$ (see [35]). For the remaining subgroups, using [9, Lemma 3.8], we have that for $Q = D_2 \times \mathbb{Z}/2 *_{D_2} D_2 \times \mathbb{Z}/2$, $\tilde{K}_0(\mathbb{Z}Q) \cong NK_0(\mathbb{Z}D_2; B_1, B_2)$, where $B_i = \mathbb{Z}[D_2 \times \mathbb{Z}/2 \setminus D_2 \times \{0\}]$ is the $\mathbb{Z}D_2$ bi-module generated by $D_2 \times \mathbb{Z}/2 \setminus D_2 \times \{0\}$ for $i = 1, 2$. For $Q = D_6 *_{D_3} D_6$, we have that $\tilde{K}_0(\mathbb{Z}Q) \cong NK_0(\mathbb{Z}D_3; C_1, C_2)$, where $C_i = \mathbb{Z}[D_6 \setminus D_3]$ is the $\mathbb{Z}D_3$-bi-module generated by $D_6 \setminus D_3$ for $i = 1, 2$. The Nil-groups NK_0 appearing in these computations are Waldhausen's Nil-groups.

We point out that the bi-modules B_i are isomorphic to the bi-module $\mathbb{Z}D_2$, and likewise, the bi-modules C_i are isomorphic to the bi-modules $\mathbb{Z}D_3$. To simplify our notation, we shall henceforth use the coefficients $\mathbb{Z}D_2, \mathbb{Z}D_3$, instead of the coefficients B_i, C_i respectively.

The computation in this section now shows that the only non-trivial term (with $q = 0$) in the spectral sequence is

$$E^2_{0,0} = \mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus NK_0(\mathbb{Z}D_2; \mathbb{Z}D_2, \mathbb{Z}D_2) \oplus NK_0(\mathbb{Z}D_3; \mathbb{Z}D_3, \mathbb{Z}D_3).$$

$q = 1$. Oliver in [30] showed that $\mathrm{Wh}(F) = 0$ when F is any of the finite subgroups that occur as stabilizers of the n-cells, for $n = 1, \ldots, 8$. For $n = 1$, we have 1-cells with stabilizers which are virtually infinite cyclic subgroups $\mathbb{Z}$ and D_∞. But again, it is well known that $\mathrm{Wh}(Q) = 0$ if $Q = \mathbb{Z}$ or D_∞ (see [3]), therefore $E^2_{p,1} = 0$ for $p \geq 1$. As before for $p = 0$ the complex may have non-zero terms in dimension zero and the resulting homology is:

$$H_0(\widehat{X}; \{\mathrm{Wh}(F_\sigma)\}) = \bigoplus_{F_{\sigma^0} \in \mathcal{FIN}} \mathrm{Wh}(F_{\sigma^0}) \oplus \bigoplus_{Q_{\sigma^0} \in \mathcal{VC}_\infty} \mathrm{Wh}(Q_{\sigma^0}).$$

If F_{σ^0} is one of the finite subgroups $\mathbb{Z}/2$, D_2, D_4, or D_6, then $\mathrm{Wh}(F) = 0$ (see [30]). If $F = \mathbb{Z}/2 \times D_4$, or $F = \mathbb{Z}/2 \times S_4$, Ortiz in [31, pg. 352] showed that $\mathrm{Wh}(\mathbb{Z}/2 \times D_4) = 0$, and $\mathrm{Wh}(\mathbb{Z}/2 \times S_4) = 0$.

If Q_{σ^0} is one of the maximal infinite virtually cyclic subgroups $\mathbb{Z}$, D_∞, $\mathbb{Z} \times \mathbb{Z}/2$, or $D_\infty \times \mathbb{Z}/2$, then $\mathrm{Wh}(Q) = 0$ (see [35]). For the remaining subgroups, using [9, Lemma 3.8], we have that for $Q = D_2 \times \mathbb{Z}/2 *_{D_2} D_2 \times \mathbb{Z}/2$, $\mathrm{Wh}(Q) \cong NK_1(\mathbb{Z}D_2; \mathbb{Z}D_2, \mathbb{Z}D_2)$, and similarly for $Q = D_6 *_{D_3} D_6$, we have that $\mathrm{Wh}(Q) \cong NK_1(\mathbb{Z}D_3; \mathbb{Z}D_3, \mathbb{Z}D_3)$. The Nil-groups NK_1 appearing in these computations are Waldhausen's Nil-groups. It follows that

$$E^2_{0,1} = NK_1(\mathbb{Z}D_2; \mathbb{Z}D_2, \mathbb{Z}D_2) \oplus NK_1(\mathbb{Z}D_3; \mathbb{Z}D_3, \mathbb{Z}D_3).$$

This completes the verification that the spectral sequence collapses at the E^2-stage, giving us the following preliminary results on the algebraic K-groups $\mathrm{Wh}_n(\Gamma_3)$ for $n < 2$.

$$\begin{aligned}
&\mathrm{Wh}(\Gamma_3) \cong NK_1(\mathbb{Z}D_2; \mathbb{Z}D_2, \mathbb{Z}D_2) \oplus NK_1(\mathbb{Z}D_3; \mathbb{Z}D_3, \mathbb{Z}D_3)\\
&\tilde{K}_0(\mathbb{Z}\Gamma_3) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus NK_0(\mathbb{Z}D_2; \mathbb{Z}D_2, \mathbb{Z}D_2) \oplus NK_0(\mathbb{Z}D_3; \mathbb{Z}D_3, \mathbb{Z}D_3)\\
&K_{-1}(\mathbb{Z}\Gamma_3) \cong \mathbb{Z} \oplus \mathbb{Z}, \quad \text{and}\\
&K_n(\mathbb{Z}\Gamma_3) \cong 0, \quad \text{for } n < -1.
\end{aligned}$$

Finally, to conclude the proof, we observe that the Waldhausen Nil-groups that appear above are isomorphic to:

$$\begin{aligned}
&NK_1(\mathbb{Z}D_2; \mathbb{Z}D_2, \mathbb{Z}D_2) \cong NK_0(\mathbb{Z}D_2; \mathbb{Z}D_2, \mathbb{Z}D_2) \cong \bigoplus_\infty \mathbb{Z}/2\\
&NK_1(\mathbb{Z}D_3; \mathbb{Z}D_3, \mathbb{Z}D_3) \cong NK_0(\mathbb{Z}D_3; \mathbb{Z}D_3, \mathbb{Z}D_3) \cong 0.
\end{aligned}$$

The computation of these Nil-groups will be carried out in the next section. □

5. Computing the Waldhausen Nil-groups

In this section we implement an approach suggested to us by F.T. Farrell, and provide explicit computations for the Waldhausen Nil-groups appearing in the previous section. This method is discussed in more detail in [24], where it is used to establish a general relationship between certain Waldhausen Nil-groups and certain Farrell Nil-groups. The results of this section can be summarized in the following two theorems:

Theorem 5.1. *For* $i = 0, 1$, $NK_i(\mathbb{Z}D_3; \mathbb{Z}D_3, \mathbb{Z}D_3) \cong 0$.

Theorem 5.2. *For* $i = 0, 1$, $NK_i(\mathbb{Z}D_2; \mathbb{Z}D_2, \mathbb{Z}D_2) \cong \bigoplus_\infty \mathbb{Z}/2$.

The proof of these theorems will be based on the following observations:

(1) There exists a *surjection*:

$$2 \cdot NK_i(\mathbb{Z}[D_k]) \to NK_i(\mathbb{Z}D_k; M_1, M_2)$$

from the direct sum of two copies of the Bass Nil-groups $NK_i(\mathbb{Z}[D_k])$ to the Waldhausen Nil-groups for $i = 0, 1$ ($k = 2, 3$). M_1, M_2 are suitably defined bi-modules that specialize to B_1, B_2 and C_1, C_2 when $k = 2, 3$ respectively (recall that in the two cases we are dealing with, we have isomorphisms of bi-modules: $B_i \cong \mathbb{Z}D_2$ and $C_i \cong \mathbb{Z}D_3$).

(2) The surjection above is an *injection* on each individual copy of $NK_i(\mathbb{Z}[D_k])$.

Let us now outline how Theorems 5.1 and 5.2 follow from these observations. In the case where $k = 3$, we know that $NK_i(\mathbb{Z}[D_3]) \cong 0$ ($i = 0, 1$), and the surjectivity of the map immediately tells us that the Waldhausen Nil-groups $NK_i(\mathbb{Z}D_3; C_1, C_2)$ vanish. On the other hand, when $k = 2$, we establish that $NK_i(\mathbb{Z}[D_2]) \cong \bigoplus_\infty \mathbb{Z}/2$ ($i = 0, 1$), an infinite countable sum of $\mathbb{Z}/2$. This is done by establishing that $NK_i(\mathbb{Z}[D_2]) \cong NK_{i+1}(\mathbb{F}_2[\mathbb{Z}/2])$ ($i = 0, 1$), and appealing to computations of Madsen [28] for the latter groups. Once we have this result for $NK_i(\mathbb{Z}[D_2])$, the surjectivity and injectivity statements above combine to force $NK_i(\mathbb{Z}D_2; B_1, B_2) \cong \bigoplus_\infty \mathbb{Z}/2$ ($i = 0, 1$).

Let us start by recalling some well-known facts about infinite virtually cyclic groups. If Γ is an infinite virtually cyclic group not of the form $G \rtimes \mathbb{Z}$, then Γ always maps epimorphically onto the infinite dihedral group D_∞ with finite kernel (see [18, Lemma 2.5]). Using this epimorphism, Farrell and Jones in [18] constructed a stratified fiber bundle $\rho_E : E \to X$, where E is a closed manifold with $\pi_1 E = \Gamma$ and such that, for each point $x \in X$, $\pi_1 \rho_E^{-1}(x) = G_x$ or $G_x \rtimes \mathbb{Z}$, where G_x is a finite group.

Next, we briefly describe the control space X as constructed by Farrell and Jones in [18, Section 2].

Recall the model for the real hyperbolic $\mathbb{H}^2$ discussed in [16, Section 2], i.e., $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid -1 < y < 1\}$, with the Riemannian metric defined in [18, pg. 23].

Let $S\mathbb{H}^2 \to \mathbb{H}^2$ be the tangent sphere bundle consisting of unit length vectors, and let $S^+\mathbb{H}^2 \to \mathbb{H}^2$ be the subbundle of $S\mathbb{H}^2 \to \mathbb{H}^2$ whose fiber over $x \in \mathbb{H}^2$ is $S_x^+\mathbb{H}^2$ (*the asymptotic northern hemisphere* defined in [16, 0.12]). The natural action of D_∞ on $\mathbb{H}^2$ determined by $D_\infty \subseteq \mathrm{Isom}(\mathbb{H}^2)$ induces an action of D_∞ on $S^+\mathbb{H}^2$.

The orbit space $S^+\mathbb{H}^2/D_\infty$ has a stratification with six strata. The lowest strata are H, V_0, V_1, the intermediate strata are B_0, B_1 and T is the top stratum. Using the canonical quotient map

$$p : S^+\mathbb{H}^2 \to S^+\mathbb{H}^2/D_\infty,$$

these strata are defined in [18, pg. 24].

Here the key point is to note that H is diffeomorphic to the circle S^1 while V_0 and V_1 are both diffeomorphic to $\mathbb{R}$. Also $p(\partial S^+\mathbb{H}^2)$ is diffeomorphic to the cylinder $S^1 \times \mathbb{R}$, and H cuts this manifold into two connected components B_0 and B_1, i.e.

$$B_0 \coprod B_1 = p(\partial S^+\mathbb{H}^2) - H.$$

Note that each pair $(B_i \cup H, H)$, $i = 0, 1$, is diffeomorphic to the pair $(S^1 \times [0, +\infty), S^1 \times 0)$. Finally, the top stratum is the complement in $S^+\mathbb{H}^2/D_\infty$ of the union $V_0 \cup V_1 \cup p(\partial S^+\mathbb{H}^2)$.

The control space X is defined to be the quotient space of $S^+\mathbb{H}^2/D_\infty$ obtained by identifying the stratum H to a single point $*$. Let

$$\rho : S^+\mathbb{H}^2/D_\infty \to X$$

denote the canonical quotient map. Then the stratification of $S^+\mathbb{H}^2/D_\infty$ induces a stratification of X whose six strata are the images of the strata in $S^+\mathbb{H}^2/D_\infty$ under ρ. Since

$$\rho : S^+\mathbb{H}^2/D_\infty - H \to X - *$$

is a homeomorphism, we identify via ρ the strata in $S^+\mathbb{H}^2/D_\infty$ different from H with the corresponding strata in X different from $*$.

This concludes the description of the control space X. For specific details and the construction of the stratified fiber bundle $\rho_E : E \to X$ mentioned earlier, we refer the reader to [18, Section 2].

Farrell and Jones also proved in [18, Theorem 2.6] that the group homomorphism

$$\pi_i(\mathcal{A}) : \pi_i \mathcal{P}_*(E; \rho_E) \longrightarrow \pi_i \mathcal{P}_*(E)$$

is an epimorphism for every integer i. Here $\mathcal{P}_*(E)$ is the spectrum of stable topological pseudo-isotopies on E, $\mathcal{P}_*(E, \rho_E)$ is the spectrum of those stable pseudo-isotopies which are controlled over X via ρ_E, and $\mathcal{A}$ is the 'assembly' map.

Recall that Quinn [36, Theorem 8.7] constructed a spectral sequence $E^n_{s,t}$, abutting to $\pi_{s+t}\mathcal{P}_*(E; \rho_E)$ with $E^2_{s,t} = H_s(X; \pi_t \mathcal{P}_*(\rho_E))$. Here $\pi_q \mathcal{P}_*(\rho_E)$, $q \in \mathbb{Z}$, denotes the stratified system of abelian groups over X where the group above $x \in X$ is $\pi_q(\mathcal{P}_*(\rho_E^{-1}(x)))$. Note that by Anderson and Hsiang's result (see [2, Theorem 3] and Section 4), $\pi_i \mathcal{P}_{*+2}(\rho_E^{-1}(x)) = K_i(\mathbb{Z}\pi_1(\rho_E^{-1}(x)))$ for $i \le -1$, $\tilde{K}_0(\mathbb{Z}\pi_1(\rho_E^{-1}(x)))$, for $i = 0$, and $\mathrm{Wh}(\pi_1(\rho_E^{-1}(x)))$ for $i = 1$.

It is important to emphasize that for each $x \in X$, the fundamental group $\pi_1 \rho_E^{-1}(x)$ of the fiber is either finite or a semidirect product $F \rtimes \mathbb{Z}$ where F is a finite subgroup of Γ (see [18, Remark 2.6.8]).

These facts together with the immediate consequences of the construction of the control space X and the stratified fiber bundle $\rho_E : E \to X$ are extremely useful in analyzing Quinn's spectral sequence. This analysis is then used to complete the proof of the *surjectivity* part in Theorems 5.1 and 5.2.

Proof of Theorem 5.1. For $\Gamma = D_3 \times D_\infty \cong D_6 *_{D_3} D_6$, first note that the fundamental group F of a fiber of ρ_E, i.e. $\pi_1 \rho_E^{-1}(x)$ is either the group $D_3 \times \mathbb{Z}$ or some finite subgroup F of Γ. It is a fact that $K_q(\mathbb{Z}F)$ (for $q \le -1$), $\tilde{K}_0(\mathbb{Z}F)$, and $\mathrm{Wh}(F)$ all vanish when F is a finite subgroup of Γ except when $F = D_6$. This fact can be obtained by combining results from [8,38,39,31,35,10,30] (see the arguments in Section 4). It is also a fact that $\tilde{K}_0(\mathbb{Z}D_6)$, $\mathrm{Wh}(D_6)$ both vanish, and $K_{-1}(\mathbb{Z}D_6) = \mathbb{Z}$ (see [31, p. 350], [35, p. 274]). To compute the lower algebraic K-theory of the ring $\mathbb{Z}[D_3 \times \mathbb{Z}] = \mathbb{Z}[D_3][\mathbb{Z}]$, we use the Fundamental Theorem of algebraic K-theory (see [3]). The following are the results found:

$$\begin{aligned}
&\mathrm{Wh}(D_3 \times \mathbb{Z}) \cong NK_1(\mathbb{Z}[D_3]) \oplus NK_1(\mathbb{Z}[D_3]),\\
&\tilde{K}_0(\mathbb{Z}[D_3 \times \mathbb{Z}]) \cong NK_0(\mathbb{Z}[D_3]) \oplus NK_0(\mathbb{Z}[D_3]),\\
&K_q(\mathbb{Z}[D_3 \times \mathbb{Z}]) \cong 0 \quad \text{for } q \le -1,
\end{aligned}$$

where the Nil-groups appearing in these computations are Bass's Nil-groups.

Next, note that given the simple nature of the control space X, the spectral sequence collapses at E^2 and for all $s, t \in \mathbb{Z}$

$$E^2_{s,t} = H_s(X; \pi_t \mathcal{P}_*(\rho_E)) \Longrightarrow \pi_{s+t}\mathcal{P}_*(E; \rho_E).$$

Combining the above with the fact that $\mathrm{Wh}_q(D_3)$ vanish for all $q \le 1$, we have that for all $s, t \in \mathbb{Z}$

$$\begin{aligned}
H_s(X; \pi_t \mathcal{P}_*(\rho_E)) &= H_s(*; \mathrm{Wh}_t(D_3 \times \mathbb{Z})) \oplus H_s(\mathbb{R}; \mathrm{Wh}_t(D_6)) \oplus H_s(\mathbb{R}; \mathrm{Wh}_t(D_6))\\
&= \mathrm{Wh}_t(D_3 \times \mathbb{Z}) \oplus \mathrm{Wh}_t(D_6) \oplus \mathrm{Wh}_t(D_6).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\pi_{-1}\mathcal{P}_*(E; \rho_E) \cong NK_1(\mathbb{Z}[D_3]) \oplus NK_1(\mathbb{Z}[D_3]),\\
&\pi_{-2}\mathcal{P}_*(E; \rho_E) \cong NK_0(\mathbb{Z}[D_3]) \oplus NK_0(\mathbb{Z}[D_3]),\\
&\pi_{-3}\mathcal{P}_*(E; \rho_E) \cong K_{-1}(\mathbb{Z}[D_3 \times \mathbb{Z}]) \oplus K_{-1}(\mathbb{Z}[D_6]) \oplus K_{-1}(\mathbb{Z}[D_6]) \cong \mathbb{Z} \oplus \mathbb{Z},\\
&\pi_q \mathcal{P}_*(E; \rho_E) = 0 \quad \text{when } q \le -4.
\end{aligned}$$

On the other hand using the results found in Section 4 for the lower algebraic K-theory of the integral group ring of $D_3 \times D_\infty$, we have that

$$\begin{aligned}
&\pi_{-1}\mathcal{P}_*(E) \cong \mathrm{Wh}(D_3 \times D_\infty) \cong NK_1(\mathbb{Z}D_3; \mathbb{Z}D_3, \mathbb{Z}D_3)\\
&\pi_{-2}\mathcal{P}_*(E) \cong \tilde{K}_0(\mathbb{Z}[D_3 \times D_\infty]) \cong NK_0(\mathbb{Z}D_3; \mathbb{Z}D_3, \mathbb{Z}D_3)\\
&\pi_{-3}\mathcal{P}_*(E) \cong K_{-1}(\mathbb{Z}[D_3 \times D_\infty]) \cong \mathbb{Z} \oplus \mathbb{Z}\\
&\pi_q\mathcal{P}_*(E) \cong K_q(\mathbb{Z}[D_3 \times D_\infty]) \cong 0 \quad \text{for } q \leq 4.
\end{aligned}$$

Here, as before, the Nil-groups appearing in the computations are Waldhausen's Nil-groups.

Combining the above with [18, Theorem 2.6] we obtain for $i = 0, 1$ the desired epimorphism

$$2 \cdot NK_i(\mathbb{Z}[D_3]) \longrightarrow NK_i(\mathbb{Z}D_3; \mathbb{Z}D_3, \mathbb{Z}D_3) \longrightarrow 0.$$

Since $NK_i(\mathbb{Z}[D_3]) = 0$ for $i = 0, 1$ (see [19]), then it follows that for $i = 0, 1$, $NK_i(\mathbb{Z}D_3; \mathbb{Z}D_3, \mathbb{Z}D_3) = 0$ completing the proof of Theorem 5.1. □

Before proving Theorem 5.2, we prove the following lemmas.

Lemma 5.3.

$$NK_0(\mathbb{Z}[D_2]) \cong NK_1(\mathbb{F}_2[\mathbb{Z}/2]) \cong \bigoplus_\infty \mathbb{Z}/2.$$

Proof. Write $\mathbb{Z}[D_2] = \mathbb{Z}[\mathbb{Z}/2 \times \mathbb{Z}/2]$ as $\mathbb{Z}[\mathbb{Z}/2][\mathbb{Z}/2]$, and consider the following Cartesian square

$$\begin{tikzcd}
\mathbb{Z}[\mathbb{Z}/2][\mathbb{Z}/2] \arrow[r] \arrow[d] & \mathbb{Z}[\mathbb{Z}/2] \arrow[d] \\
\mathbb{Z}[\mathbb{Z}/2] \arrow[r] & \mathbb{F}_2[\mathbb{Z}/2].
\end{tikzcd}$$

This Cartesian square yields the Mayer–Vietoris sequence

$$\begin{aligned}
\cdots &\to NK_1(\mathbb{Z}[\mathbb{Z}/2]) \oplus NK_1(\mathbb{Z}[\mathbb{Z}/2]) \to NK_1(\mathbb{F}_2[\mathbb{Z}/2]) \to NK_0(\mathbb{Z}[\mathbb{Z}/2][\mathbb{Z}/2])\\
&\to NK_0(\mathbb{Z}[\mathbb{Z}/2]) \oplus NK_0(\mathbb{Z}[\mathbb{Z}/2]) \to \cdots.
\end{aligned}$$

Since $NK_i(\mathbb{Z}[\mathbb{Z}/2]) = 0$, for $i = 0, 1$ (see [19]), we obtain that $NK_0(\mathbb{Z}[D_2]) \cong NK_1(\mathbb{F}_2[\mathbb{Z}/2])$. In [28], Madsen showed that $NK_1(\mathbb{F}_2[\mathbb{Z}/2]) \cong \bigoplus_\infty \mathbb{Z}/2$, giving us the desired result. □

Lemma 5.4.

$$NK_1(\mathbb{Z}[D_2]) \cong NK_2(\mathbb{F}_2[\mathbb{Z}/2]) \cong \bigoplus_\infty \mathbb{Z}/2.$$

Proof. As before, consider the following Cartesian square

$$\begin{tikzcd}
\mathbb{Z}[\mathbb{Z}/2][\mathbb{Z}/2] \arrow[r] \arrow[d] & \mathbb{Z}[\mathbb{Z}/2] \arrow[d] \\
\mathbb{Z}[\mathbb{Z}/2] \arrow[r] & \mathbb{F}_2[\mathbb{Z}/2].
\end{tikzcd}$$

As mentioned in Lemma 5.3, since $NK_1(\mathbb{Z}[\mathbb{Z}/2])$ vanishes, the Mayer–Vietoris sequence yields the epimorphism

$$NK_2(\mathbb{F}_2[\mathbb{Z}/2]) \to NK_1(\mathbb{Z}[\mathbb{Z}/2][\mathbb{Z}/2]) \longrightarrow 0.$$

Now, in [28], Madsen has shown that $NK_2(\mathbb{F}_2[\mathbb{Z}/2]) \cong \bigoplus_\infty \mathbb{Z}/2$. Combining this result with the well-known fact that the Nil-groups are either trivial or infinitely generated [14], we conclude that up to isomorphism, either

$$NK_1(\mathbb{Z}[D_2]) \cong \bigoplus_\infty \mathbb{Z}/2$$

or the group is trivial. But the group $NK_1(\mathbb{Z}[\mathbb{Z}/2][\mathbb{Z}/2])$ is known to be non-trivial (see [3]), completing the proof of the lemma. □

Proof of Theorem 5.2. Let $\Gamma = D_2 \times D_\infty \cong D_2 \times \mathbb{Z}/2 *_{D_2} D_2 \times \mathbb{Z}/2$. In this case note that the fundamental group F of a fiber of ρ_E, i.e. $\pi_1\rho_E^{-1}(x)$ is either the group $D_2 \times \mathbb{Z}$ or some finite subgroup F of Γ. It is a fact that $K_i(\mathbb{Z}F)$ (for $i \le -1$), $\tilde{K}_0(\mathbb{Z}F)$, and $\mathrm{Wh}(F)$ all vanish when F is a finite subgroup of Γ. This fact can be obtained by combining results from [8,38,39,31,35,10,30] (see Section 4). To compute the lower algebraic K-theory of the ring $\mathbb{Z}[D_2 \times \mathbb{Z}] = \mathbb{Z}[D_2][\mathbb{Z}]$, we apply the Fundamental Theorem of algebraic K-theory to obtain:

$$\mathrm{Wh}(D_2 \times \mathbb{Z}) \cong NK_1(\mathbb{Z}[D_2]) \oplus NK_1(\mathbb{Z}[D_2])$$
$$\tilde{K}_0(\mathbb{Z}[D_2 \times \mathbb{Z}]) \cong NK_0(\mathbb{Z}[D_2]) \oplus NK_0(\mathbb{Z}[D_2])$$
$$K_q(\mathbb{Z}[D_2 \times \mathbb{Z}]) \cong 0 \quad \text{for } q \le -1,$$

where the Nil-groups appearing in these computations are Bass's Nil-groups.

Once again, given the simple nature of the control space X in this case, the spectral sequence collapses at E^2 and for all $s, t \in \mathbb{Z}$

$$E^2_{s,t} = H_s(X; \pi_t\mathcal{P}_*(\rho_E)) \Longrightarrow \pi_{s+t}\mathcal{P}_*(E; \rho_E).$$

Combining the above with the fact that $\mathrm{Wh}_q(D_2)$ and $\mathrm{Wh}_q(D_2 \times \mathbb{Z}/2)$ are both trivial for all $q \le 1$, we have that for all $s, t \in \mathbb{Z}$

$$H_s(X; \pi_t\mathcal{P}_*(\rho_E)) = \mathrm{Wh}_t(D_2 \times \mathbb{Z}).$$

Consequently, we obtain:

$$\pi_{-1}\mathcal{P}_*(E; \rho_E) \cong NK_1(\mathbb{Z}[D_2]) \oplus NK_1(\mathbb{Z}[D_2]),$$
$$\pi_{-2}\mathcal{P}_*(E; \rho_E) \cong NK_0(\mathbb{Z}[D_2]) \oplus NK_0(\mathbb{Z}[D_2]),$$
$$\pi_q\mathcal{P}_*(E; \rho_E) = 0 \quad \text{when } q \le -3.$$

On the other hand, using the results found in Section 4 for the lower algebraic K-theory of $D_2 \times D_\infty$, we have that

$$\pi_{-1}\mathcal{P}_*(E) \cong \mathrm{Wh}(D_2 \times D_\infty) \cong NK_1(\mathbb{Z}D_2; \mathbb{Z}D_2, \mathbb{Z}D_2)$$
$$\pi_{-2}\mathcal{P}_*(E) \cong \tilde{K}_0(\mathbb{Z}[D_2 \times D_\infty]) \cong NK_0(\mathbb{Z}D_2; \mathbb{Z}D_2, \mathbb{Z}D_2)$$
$$\pi_q\mathcal{P}_*(E) \cong K_i(\mathbb{Z}[D_2 \times D_\infty]) \cong 0 \quad \text{for } q \le -3.$$

Recall that the Nil-groups appearing in these computations are Waldhausen's Nil-groups.

Combining the above with [18, Theorem 2.6] we obtain for $i = 0, 1$ the epimorphism

$$2 \cdot NK_i(\mathbb{Z}[D_2]) \longrightarrow NK_i(\mathbb{Z}D_2; \mathbb{Z}D_2, \mathbb{Z}D_2) \longrightarrow 0.$$

Next, recall that a monomorphism of groups $\sigma : \Pi \to \Gamma$ induces transfer maps $\sigma^* : \mathrm{Wh}(\Gamma) \to \mathrm{Wh}(\Pi)$ and $\sigma^* : \tilde{K}_0(\mathbb{Z}\Gamma) \to \tilde{K}_0(\mathbb{Z}\Pi)$ provided the image of σ has finite index in Γ (see [15]).

Let $\Pi = G \times \mathbb{Z}$, where $G = D_2$. By the above paragraph, the monomorphism σ in the exact sequence

$$0 \longrightarrow \Pi \xrightarrow{\sigma} \Gamma \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

induces the transfer maps:

$$\sigma^* : \mathrm{Wh}(G \times D_\infty) \longrightarrow \mathrm{Wh}(G \times \mathbb{Z})$$
$$\sigma^* : \tilde{K}_0(\mathbb{Z}[G \times D_\infty]) \longrightarrow \tilde{K}_0(\mathbb{Z}[G \times \mathbb{Z}]).$$

Now observe that, from the short exact sequence above we have an induced action of the non-trivial element $\tau \in \mathbb{Z}/2$ on the kernel $\Pi = G \times \mathbb{Z}$, inducing a map τ_* on the K-groups of $G \times \mathbb{Z}$. Furthermore, the action of this element τ on $G \times \mathbb{Z}$ sends $(g, t) \mapsto (g, t^{-1})$ (where t is a generator for $\mathbb{Z}$). This implies that in the direct sum decomposition given by the *Fundamental Theorem of algebraic K-theory* of the algebraic K-groups of the ring $\mathbb{Z}[G][\mathbb{Z}]$:

$$K_i(\mathbb{Z}[G][\mathbb{Z}]) \cong K_{i-1}(\mathbb{Z}[G]) \oplus K_i(\mathbb{Z}[G]) \oplus NK_i(\mathbb{Z}[G]) \oplus NK_i(\mathbb{Z}[G]),$$

τ_* interchanges the two direct summands $NK_i(\mathbb{Z}[G])$. Now recall that $(\sigma^* \circ \sigma_*)(x) = x + \tau_*(x)$ for all $x \in \mathrm{Wh}_i(D_2 \times \mathbb{Z})$, $i = 0, 1$; we will denote this map by ψ. This yields the following commutative diagram for $i = 0, 1$:

$$\begin{tikzcd}
\mathrm{Wh}_i(D_2 \times \mathbb{Z}) \arrow[r, "\sigma_*"] \arrow[dr, "\psi"'] & \mathrm{Wh}_i(D_2 \times D_\infty) \arrow[d, "\sigma^*"] \\
& \mathrm{Wh}_i(D_2 \times \mathbb{Z}).
\end{tikzcd}$$

Using our earlier computations, we obtain for $i = 0, 1$ the commutative diagram:

$$\begin{tikzcd}
NK_i(\mathbb{Z}[D_2]) \oplus NK_i(\mathbb{Z}[D_2]) \arrow[r, "\sigma_*"] \arrow[dr, "\psi"'] & NK_i(\mathbb{Z}D_2; \mathbb{Z}D_2, \mathbb{Z}D_2) \arrow[d, "\sigma^*"] \\
& NK_i(\mathbb{Z}[D_2]) \oplus NK_i(\mathbb{Z}[D_2]).
\end{tikzcd}$$

We now define the map φ by:

$$\varphi : NK_i(\mathbb{Z}[D_2]) \xrightarrow{(\mathrm{Id}, 0)} NK_i(\mathbb{Z}[D_2]) \oplus NK_i(\mathbb{Z}[D_2]).$$

Note that $\sigma^* \circ \sigma_*$ is injective on $\varphi(NK_i(\mathbb{Z}[D_2]))$. It follows that σ_* is injective on the first summand of the direct sum $NK_i(\mathbb{Z}[D_2]) \oplus NK_i(\mathbb{Z}[D_2])$. Note that an identical argument can be used to show that $\sigma^* \circ \sigma_*$ is injective on the second summand (though it is easy to see that the image of the first and the second summand coincide).

For $i = 0, 1$, this gives us the desired monomorphism:

$$0 \longrightarrow NK_i(\mathbb{Z}[D_2]) \longrightarrow NK_i(\mathbb{Z}D_2; \mathbb{Z}D_2, \mathbb{Z}D_2).$$

Combining this monomorphism with the epimorphism found earlier, and applying Lemmas 5.3 and 5.4, it follows that $i = 0, 1$

$$NK_i(\mathbb{Z}D_2; B_1, B_2) \cong \bigoplus_{\infty} \mathbb{Z}/2.$$

This completes the proof of Theorem 5.2. $\square$

Combining the computations of the Nil-groups contained in this section with the formula at the end of the previous section completes the proof of Theorem 1.1.

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Appendix. The Alves–Ontaneda construction for $P4m$

In this appendix, we briefly summarize the Alves–Ontaneda construction of a finite dimensional classifying space (with isotropy in $\mathcal{VC}$) for the 2-dimensional crystallographic group $P4m$. We start with an outline of their general construction for classifying spaces of crystallographic groups, then specialize to the case of $P4m$, and finally, detail the computation of the isotropy groups of cells in each dimension.

We start by recalling that an n-dimensional crystallographic group Γ fits into a short exact sequence of the form:

$$0 \to T \to \Gamma \to F \to 0$$

where $T \cong \mathbb{Z}^n$, and F is a finite subgroup of the orthogonal group $O(n)$. For the natural action of Γ on $\mathbb{R}^n$, the subgroup T is the subgroup of Γ that acts via translations of $\mathbb{R}^n$, while the quotient F encodes the "rotational part" of the Γ-action. One now considers the space $C(n)$ obtained by quotienting $\mathbb{R}^n$ by the equivalence relation $v \sim \pm v$; this space is naturally identified with the open cone over the real projective space $\mathbb{R}P^{n-1}$, with vertex point identified with the equivalence class of the origin of $\mathbb{R}^n$. Note that there is a Γ-action on $C(n)$, induced from the projection $\Gamma \to F \leq O(n)$ via the natural action of $O(n)$ on $C(n)$.

The model $E_{\mathcal{VC}}(\Gamma)$ constructed by Alves–Ontaneda is defined to be a suitable quotient of $C(n) \times \mathbb{R}^n$, equipped with the diagonal Γ-action. The idea is the following: the subgroup T acts by translations on $\mathbb{R}^n$, and for each translation we can consider the corresponding vector $h \in \mathbb{R}^n$. One then defines the following:

- l_h to be the line from the origin in the direction h.
- $\mathcal{L}_h$ to be the collection of all lines in $\mathbb{R}^n$ parallel to l_h.

- $\mathcal{L}$ to be the union of all the sets $\mathcal{L}_h$, ranging over all translations arising in the T-action on $\mathbb{R}^n$.

Now there are natural Γ-actions on $\mathcal{L}$ (via the Γ-action on $\mathbb{R}^n$). For each line $l \in \mathcal{L}_h$, one takes an appropriate "lift" $c(l) \times l \subset C(n) \times \mathbb{R}^n$, where $c(l) \in C(n)$ is suitably defined, and depends solely on the set $\mathcal{L}_h$ containing l. One then collapses all these "lifted lines" down to points. The net effect is that, in the space $C(n) \times \mathbb{R}^n$, one has collapsed down, at various points $c(l) \in C(n)$, the copy of $\mathbb{R}^n$ to a copy of $\mathbb{R}^{n-1}$ which is "transverse" to the line l. We refer the reader to [1, Section 2] for more details concerning the construction, as well as for the proof that the resulting space is a model for $E_{\mathcal{VC}}(\Gamma)$.

We now specialize to the case where $\Gamma = P4m$, the 2-dimensional crystallographic group that appears as the cusp group in Γ_3. In this case, we have that the model $E_{\mathcal{VC}}(\Gamma)$ is a quotient of $C(2) \times \mathbb{R}^2$, a 4-dimensional space. We let $Z \subset E_{\mathcal{VC}}(\Gamma)$ consist of the set of all points which are obtained by collapsing one of the sets $c(l) \times l \subset C(2) \times \mathbb{R}^2$. Note that the subset Z consists of a union of 1-dimensional subsets of $E_{\mathcal{VC}}(\Gamma)$, and that on the complement $E_{\mathcal{VC}}(\Gamma) - Z$, the quotient map

$$C(2) \times \mathbb{R}^2 \to E_{\mathcal{VC}}(\Gamma) - Z$$

is a homeomorphism. We now proceed to detail the isotropy subgroups for the cells appearing in the space $E_{\mathcal{VC}}(\Gamma)$, for the Γ-equivariant CW-structure defined in [1, Section 2.2]. Observe that a cell contained in Z can only be of dimension 0 or 1, while cells that lie in $E_{\mathcal{VC}}(\Gamma) - Z$ decomposes naturally as a product of cells in $C(2)$ with cells in $\mathbb{R}^2$.

Stabilizers of 0-*cells*: There are two possibilities: either the 0-cell σ^0 lies in Z, or it lies in the complement of Z.

- if σ^0 lies in Z, and Stab(σ^0) $\leq \Gamma$ fixes σ^0, then Stab(σ^0) must stabilize the corresponding line $l \subset \mathbb{R}^2$ (whose quotient gives σ^0). The restriction of the action of Stab(σ^0) to the line l is either via $\mathbb{Z}$ or via D_∞. In the direction transverse to l, the subset Stab(σ^0) can either act trivially, or reverse the transverse direction, allowing an additional $\mathbb{Z}/2$ factor. Hence the possible isotropy groups are either $\mathbb{Z}$, D_∞, $\mathbb{Z} \times \mathbb{Z}/2$, or $D_\infty \times \mathbb{Z}/2$.
- if σ^0 lies in $E_{\mathcal{VC}}(\Gamma) - Z$, then one can write σ^0 as a product $\sigma^0_C \times \sigma^0_R$, where $\sigma^0_C \subset C(2)$ and $\sigma^0_R \subset \mathbb{R}^2$. In particular, the subset Stab(σ^0) must fix the point $\sigma^0_R \in \mathbb{R}^2$, and hence must be isomorphic to a subgroup of a point stabilizer for the Γ-action on $\mathbb{R}^2$. Furthermore, if $\sigma^0_C \neq [0] \in C(2)$, then the group must additionally stabilize a line l through the point (corresponding to the direction σ^0_C), forcing the group to be either 1, $\mathbb{Z}/2$, or D_2 according to whether it preserves/reverses orientation on l and $l^\perp$. On the other hand, if $\sigma^0_C = [0] \in C(2)$, then we have no additional constraints, and Stab(σ^0) = Stab(σ^0_R), hence must be either 1, $\mathbb{Z}/2$, D_2, or D_4. Combining these two cases, we see that the only possible isotropy groups are 1, $\mathbb{Z}/2$, D_2, or D_4.

Stabilizer of 1-*cells*: There are again two possibilities, according to whether the 1-cell σ^1 lies in Z or lies in the complement of Z.

- if σ^1 lies in Z, and Stab(σ^1) $\leq \Gamma$ fixes σ^1, then Stab(σ^1) must stabilize a 1-parameter family of parallel lines in $\mathbb{R}^2$ (whose quotient gives σ^1). Hence the subgroup Stab(σ^1) is either $\mathbb{Z}$ or D_∞, according to the nature of the cocompact action of the group on one of the stabilized lines. Note that in this case, one *cannot* have a "transverse reflection" as could occur in the case of the stabilizer of a 0-cell in Z.
- if σ^1 lies in $E_{\mathcal{VC}}(\Gamma) - Z$, then one can write σ^1 as a product of cells in $C(2)$ and $\mathbb{R}^2$. There are two possibilities: either (1) $\sigma^1 = \sigma^0_C \times \sigma^1_R$, or (2) $\sigma^1 = \sigma^1_C \times \sigma^0_R$. In case (1), we see that Stab(σ^1) must

fix pointwise an interval $\sigma_R^1 \subset \mathbb{R}^2$, and hence must fix pointwise the line through σ_R^1. Hence the only freedom is in the transverse direction, and $\mathrm{Stab}(\sigma^1)$ must be isomorphic to either 1 or $\mathbb{Z}/2$ (according to whether the group preserves or reverses the orientation of the transverse line). In case (2), we see that $\mathrm{Stab}(\sigma^1)$ fixes the point σ_R^0 in $\mathbb{R}^2$, and stabilizes a line l through σ_R^0 (corresponding to a point in σ_C^1 distinct from $[0] \in C(2)$). Hence the group $\mathrm{Stab}(\sigma^1)$ is completely determined by whether it preserves or reverses orientation on the line l and on the line $l^\perp$. This forces $\mathrm{Stab}(\sigma^1)$ to be either 1, $\mathbb{Z}/2$, or D_2.

Stabilizer of 2-*cells*: Since the subset Z is 1-dimensional, the 2-cell σ^2 *must* be contained in $E_{\mathcal{VC}}(\Gamma)-Z$. There are three possibilities for the decomposition of σ^2 as a product: either (1) $\sigma^2 = \sigma_C^0 \times \sigma_R^2$, or (2) $\sigma^2 = \sigma_C^1 \times \sigma_R^1$, or (3) $\sigma^2 = \sigma_C^2 \times \sigma_R^0$. We analyze each of the three cases separately.

- if $\sigma^2 = \sigma_C^0 \times \sigma_R^2$, then the group $\mathrm{Stab}(\sigma^2)$ fixes pointwise a 2-cell in $\mathbb{R}^2$, and hence must be the identity.
- if $\sigma^2 = \sigma_C^1 \times \sigma_R^1$, then the group $\mathrm{Stab}(\sigma^2)$ fixes pointwise the 1-cell σ_R^1 in $\mathbb{R}^2$, and hence must fix pointwise the line l containing σ_R^1. Furthermore, the group $\mathrm{Stab}(\sigma^2)$ must stabilize at least one line l', corresponding to a point in σ_C^1 distinct from $[0] \in C(2)$. If l' coincides with l or with $l^\perp$, then the group $\mathrm{Stab}(\sigma^2)$ is either 1 or $\mathbb{Z}/2$. On the other hand, if there is an l' corresponding to a direction distinct from l or $l^\perp$, then this forces $\mathrm{Stab}(\sigma^2)$ to be the trivial group 1.
- if $\sigma^2 = \sigma_C^2 \times \sigma_R^0$, then the group $\mathrm{Stab}(\sigma^2)$ fixes the point σ_R^0 in $\mathbb{R}^2$, and fixes an open set of directions through the point σ_R^0. Recall that $C(2)$ is the cone over the projective space, hence "fixing a direction" allows you to *reverse* the orientation on a line in the corresponding direction. In particular, a rotation by an angle of π in the point σ_r^0 acts trivially on the $C(2) \times \sigma_r^0$ factor lying above the point σ_r^0, and this is the only non-trivial element in the stabilizer of σ_R^0 which fixes an open set of directions through σ_R^0. Hence the only possibilities for $\mathrm{Stab}(\sigma^2)$ is the trivial group or the group $\mathbb{Z}/2$.

Stabilizer of 3-*cells*: Since the subset Z is 1-dimensional, the 3-cell σ^3 *must* be contained in $E_{\mathcal{VC}}(\Gamma)-Z$. Since the two factors in $C(2) \times \mathbb{R}^2$ are 2-dimensional, there are two possibilities for the decomposition of σ^3 as a product: either (1) $\sigma^3 = \sigma_C^1 \times \sigma_R^2$, or (2) $\sigma^3 = \sigma_C^2 \times \sigma_R^1$. We analyze each of the two cases separately.

- if $\sigma^3 = \sigma_C^1 \times \sigma_R^2$, then the group $\mathrm{Stab}(\sigma^3)$ fixes an open set σ_R^2 of points in $\mathbb{R}^2$, and hence $\mathrm{Stab}(\sigma^3)$ must be trivial.
- if $\sigma^3 = \sigma_C^2 \times \sigma_R^1$, then the group $\mathrm{Stab}(\sigma^3)$ fixes the 1-cell $\sigma_R^1 \subset \mathbb{R}^2$ pointwise, and hence must fix the line l through the cell σ_R^1. Furthermore, $\mathrm{Stab}(\sigma^3)$ must fix an open set of directions, which by our previous discussion means that there is at most one non-trivial element in $\mathrm{Stab}(\sigma^3)$, and this must be a rotation around a fixed point by an angle of π. But this cannot occur, since $\mathrm{Stab}(\sigma^3)$ fixes the line l pointwise. We conclude that $\mathrm{Stab}(\sigma^3)$ must be trivial.

Stabilizer of 4-*cells*: We note that σ^4 must factor as a product $\sigma_C^2 \times \sigma_R^2$. In particular, $\mathrm{Stab}(\sigma^4)$ must fix an open set σ_R^2 of points in $\mathbb{R}^2$, and hence must be trivial.

This concludes the computation of cell stabilizers for $E_{\mathcal{VC}}(P4m)$, and justifies the list of stabilizers appearing in Section 4 of the present paper.

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