The Lower Algebraic K-theory of Γ_3

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Abstract. We explicitly compute the lower algebraic K-theory of Γ_3 a discrete subgroup of the group of isometries of hyperbolic 3-space.

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1. Introduction

In this paper we prove the following theorem:

MAIN THEOREM 1. *Let* $\Gamma_3 = O^+(3, 1) \cap GL(4, \mathbb{Z})$. *Then the lower algebraic K-theory of the integral group ring of Γ_3 is given as follows:*

$$Wh(\Gamma_3) = 0,$$
$$\tilde{K}_0(\mathbb{Z}\Gamma_3) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/4,$$
$$K_{-1}(\mathbb{Z}\Gamma_3) \cong \mathbb{Z} \oplus \mathbb{Z},$$
$$K_n(\mathbb{Z}\Gamma_3) = 0 \quad for \, n < -1.$$

For an arbitrary discrete group Γ, it has been conjecture that the algebraic K-theory of the integral group ring $\mathbb{Z}\Gamma$ may be computed from the corresponding K-groups of certain subgroups of Γ. More precisely, the Farrell and Jones Isomorphism Conjecture [16] states that the algebraic K-theory of $\mathbb{Z}\Gamma$ may be computed from the algebraic K-theory of the virtually cyclic subgroups of Γ (where a group is called *virtually cyclic* if it has a cyclic subgroup of finite index) via an appropriate "assembly map" (see Section 2 for a precise statement and definitions). In [16] Farrell and Jones proved the Isomorphism Conjecture in lower algebraic K-theory for cocompact discrete subgroups of a virtually connected Lie group, in particular for discrete groups acting properly discontinuously and cocompactly by isometries on a simply connected symmetric Riemannian manifold M with sectional curvature non-positive everywhere. In [5] Berkove, Farrell,

Pineda and Pearson extend this result to discrete groups acting properly discontinuously on hyperbolic n-space via isometries whose orbit space has finite volume (but non-necessarily compact).

Let $\Gamma_n = O^+(n, 1) \cap GL(n+1, \mathbb{Z})$, where $O^+(n, 1)$ denotes the group of isometries of the Riemannian manifold $\mathbb{H}^n$. The group Γ_n is a discrete subgroup of $O^+(n, 1)$, since Γ_n is a subgroup of the discrete group $GL(n+1, \mathbb{Z})$. The groups Γ_n are hyperbolic, non-cocompact, n-simplex, reflection groups for $n = 3, \ldots, 9$ (see Section 3). The groups Γ_n form a nice family of infinite groups with torsion for which the Isomorphism Conjecture in algebraic K-theory holds. In this paper we compute the lower algebraic K-theory of the integral group ring $\mathbb{Z}\Gamma_3$. We accomplish our computations using the fundamental results of Farrell and Jones [16] and the algebraic techniques of Davis and Lück [15].

2. Review of the Isomorphism Conjecture in *K*-Theory

In this section we introduce some notation that is used throughout this paper. The definitions and results provided here are brief, and the interested reader should refer to [15].

Let Γ be a discrete group and $\mathcal{F}$ be a family of subgroups of Γ closed under inclusion and conjugation, i.e., if $H \in \mathcal{F}$ then $g^{-1}H'g \in \mathcal{F}$ for all $H' \subset H$ and all $g \in \Gamma$. Some examples for $\mathcal{F}$ are $\mathcal{TR}$, $\mathcal{FIN}$, $\mathcal{VC}$, and $\mathcal{ALL}$, which are the families consisting of the trivial group, finite subgroups, virtually cyclic groups, and all subgroups, respectively.

The *orbit category* $\mathrm{Or}(\Gamma)$ is the category whose objects are homogeneous Γ-spaces Γ/H, considered as left Γ-sets, and whose morphisms are Γ-maps. More generally, for a family of subgroups $\mathcal{F}$, define the *restricted orbit category* $\mathrm{Or}(\Gamma, \mathcal{F})$ to be the category whose objects are homogeneous spaces Γ/H, where $H \in \mathcal{F}$. If $\mathcal{F}$ is the family $\mathcal{ALL}$, we abbreviate $\mathrm{Or}(\Gamma, \mathcal{ALL})$ by $\mathrm{Or}(\Gamma)$.

A *covariant (contravariant)* $\mathrm{Or}(\Gamma)$-*space X* is a covariant (contravariant) functor

$$X : \mathrm{Or}(\Gamma) \longrightarrow \mathcal{SPACES}.$$

from $\mathrm{Or}(\Gamma)$ to the category of compactly generated spaces. A *map* between $\mathrm{Or}(\Gamma)$-spaces is a natural transformation of such functors. A covariant (contravariant) $\mathrm{Or}(\Gamma, \mathcal{TR})$-space is the same as a left (right) Γ-space. Maps of $\mathrm{Or}(\Gamma, \mathcal{TR})$-spaces correspond to Γ-maps.

A *spectrum* $\mathbb{E} = \{(E(n), \sigma(n)) \,|\, n \in \mathbb{Z}\}$ is a sequence of based spaces $\{E(n) \,|\, n \in \mathbb{Z}\}$ together with pointed maps $\sigma(n) : (E(n) \wedge S^1) \to E(n+1)$, called structure maps. A spectrum $\mathbb{E}$ is called an Ω-*spectrum* if for each structure map, its adjoint $E_n \to \Omega E_{n+1} = \mathrm{map}(S^1, E_{n+1})$ is a weak homotopy equivalence of spaces. We denote by Ω-$\mathcal{SPECTRA}$ the corresponding

full subcategory of $\mathcal{SPECTRA}$. A *map* of spectra is a sequence of maps which strictly commute with the structure maps in an obvious sense.

The homotopy groups of a spectrum $\mathbb{E}$ are defined by

$$\pi_q(\mathbb{E}) := \lim_{n \to \infty} \pi_{q+n}(E(n)),$$

where the system $\pi_{q+n}(E(n))$ is given by the composition

$$\pi_{q+n}(E(n)) \xrightarrow{\Sigma} \pi_{q+n+1}(E(n) \wedge S^1) \xrightarrow{(\sigma_n)_*} \pi_{q+n+1}(E(n+1))$$

of the suspension homomorphism and the homomorphism induced by the structure maps.

More generally, one can also take the homology of a space with coefficients in a spectrum by

$$H_q(Y; \mathbb{E}) := \lim_{n \to \infty} \pi_{q+n}(Y_+ \wedge E(n)),$$

letting Y be a point recovers the coefficient groups. Homology with coefficients in a specified spectrum is a generalized homology theory; furthermore any generalized homology theory has a spectrum giving rise to it.

Associated to each covariant $\mathrm{Or}(\Gamma)$-spectrum $\mathbb{K}(?)$ (we use a question mark to indicate the place where objects are plugged into the functor) is a generalized equivariant homology theory satisfying the WHE-axiom and the disjoint union axiom, for example, there is a long exact Mayer–Vietoris sequence, and the WHE-axiom requires that weak homotopy equivalence of contravariant $\mathrm{Or}(\Gamma)$-spaces induces an isomorphism on homology groups.

This generalized homology theory is constructed as follows: every Γ-space X gives rise to a contravariant $\mathrm{Or}(\Gamma)$-space $X^? = \mathrm{map}_\Gamma(\Gamma/?, X)$ (see [15, Example 1.3]) and we can form the balanced smash product over the orbit category between a contravariant $\mathrm{Or}(\Gamma)$-space and a covariant $\mathrm{Or}(\Gamma)$-spectrum to obtain an ordinary spectrum, e.g.,

$$X_+^? \wedge_{\mathrm{Or}(\Gamma)} \mathbb{K}(?) = \left(\bigvee_{\Gamma/H \in \mathrm{Or}(\Gamma)} X_+^H \wedge \mathbb{K}(H) \right) \Big/ \sim .$$

Compare [15, p. 237]. This construction is functorial in X and satisfies the properties listed above. To stress the homological behavior we write the homotopy groups of the spectrum as

$$H_n^{\mathrm{Or}(\Gamma)}(X; \mathbb{K}) = \pi_n(X_+^? \wedge_{\mathrm{Or}(\Gamma)} \mathbb{K}(?)).$$

Note that if $\mathcal{C}$ is a category with a single object, all whose morphims are isomorphisms (e.g., $\mathrm{Or}(\Gamma, \mathcal{TR})$), this generalized homology theory reduces to Borel homology.

Let $\mathcal{C}$ be a (small) additive category. In [27] the authors construct a non-connective spectrum whose homotopy groups are the algebraic K-groups of $\mathcal{C}$ (including the negative groups). We denote this spectrum by $\mathbb{K}^{-\infty}(\mathcal{C})$. In fact $\mathbb{K}^{-\infty}$ is a functor from additive categories to $\mathcal{SPECTRA}$, i.e., an additive and hence exact functor induces a map of spectra.

Let us now recall the construction of the algebraic K-theory Or $(\Gamma, \mathcal{F})$-spectrum defined by Davis and Lück in [15]. We will denote it by $\mathbb{K}R^{-\infty}$ with R an arbitrary associative ring with unit. Given any Γ-set S define the *transport category* $\bar{S}$ the category whose objects are the elements of S and $\mathrm{mor}(s, t) = \{g \in \Gamma \mid gs = t\}$. The transport category is a groupoid, i.e., every morphism is an isomorphism. Given any small category $\mathcal{C}$, we can form the associated R-linear category $R\mathcal{C}$ with the same objects and new morphism set $\mathrm{mor}_{R\mathcal{C}}(c, d) = R\,\mathrm{mor}_{\mathcal{C}}(c, d)$ (the free R-module generated by the old morphism set). Finally we turn $R\mathcal{C}$ into an additive category, i.e., we artificially introduce finite sums (or products). The resulting category is denoted by $R\mathcal{C}_{\oplus}$. The Davis–Lück functor is now given as

$$\mathbb{K}R^{-\infty} : \mathrm{Or}(\Gamma, \mathcal{F}) \longrightarrow \mathcal{SPECTRA}$$
$$\Gamma/H \longmapsto \mathbb{K}^{-\infty}(R\overline{\Gamma/H}_{\oplus}).$$

Note that $\mathbb{K}R^{-\infty}$ and $\mathbb{K}^{-\infty}(R)$ are different objects.

For any quotient, the category $\overline{\Gamma/H}$ is equivalent to the category $\mathrm{Or}(\Gamma, \mathcal{TR})$, as a consequence the $\mathrm{Or}(\Gamma)$-spectrum defined above has the key property that $\pi_n(\mathbb{K}R^{-\infty}(\Gamma/H)) = K_n(RH)$.

For a family $\mathcal{F}$ of subgroups of Γ we denote by $E(\Gamma, \mathcal{F})$ the universal space among Γ-spaces with isotropy in $\mathcal{F}$. It is characterized by the universal property that for every Γ-CW complex X whose isotropy groups are all in $\mathcal{F}$ one can find an equivariant continuous map $X \to E(\Gamma, \mathcal{F})$ which is unique up to equivariant homotopy. A Γ-CW-complex E is a model $E(\Gamma, \mathcal{F})$ if the H-fixed point sets E^H are contractible for all $H \in \mathcal{F}$ and empty otherwise. The two extreme cases are $\mathcal{F} = \mathcal{ALL}$, where $E(\Gamma, \mathcal{F})$ can be taken to be a point, and $\mathcal{F} = \mathcal{TR}$, where $E(\Gamma, \mathcal{F})$ is a model for $E\Gamma$.

The projection map $\mathrm{pr} : E(\Gamma, \mathcal{F}) \longrightarrow E(\Gamma, \mathcal{ALL}) = \{pt\}$ induces a map

$$H_n^{\mathrm{Or}(\Gamma)}(E(\Gamma, \mathcal{F}); \mathbb{K}R^{-\infty}) \longrightarrow H_n^{\mathrm{Or}(\Gamma)}(\{pt\}; \mathbb{K}R^{-\infty}) = K_n(R\Gamma),$$

which is called the *assembly map*.

The *Isomorphism Conjecture in algebraic K-theory* for a discrete group Γ and a family $\mathcal{F}$ of subgroups is that the assembly map

$$H_n^{\mathrm{Or}(\Gamma)}(E(\Gamma, \mathcal{F}); \mathbb{K}R^{-\infty}) \longrightarrow H_n^{\mathrm{Or}(\Gamma)}(\{pt\}; \mathbb{K}R^{-\infty}) = K_n(R\Gamma)$$

is an isomorphism for all $n \in \mathbb{Z}$.

It is clear that for an arbitrary Γ the Isomorphism Conjecture need not be valid. However, the Isomorphism Conjecture is always true (and therefore pointless!) when $\mathcal{F}$ is the family of all subgroups. The philosophy is that the smaller the family, the easier it is to compute $H_n^{\mathrm{Or}(\Gamma)}(E(\Gamma, \mathcal{F}); \mathbb{K}R^{-\infty})$. The larger the family, the closer the end result is to K-theory.

The *Farrell and Jones Isomorphism Conjecture in algebraic K-theory*, reformulated in terms of the Davis and Lück functor, states that the assembly map

$$H_n^{\mathrm{Or}(\Gamma)}(E(\Gamma, \mathcal{VC}); \mathbb{K}R^{-\infty}) \longrightarrow H_n^{\mathrm{Or}(\Gamma)}(\{pt\}; \mathbb{K}R^{-\infty}) = K_n(R\Gamma)$$

is an isomorphism for all $n \in \mathbb{Z}$.

The main point of the validity of this conjecture is that it allows the computations of the groups of interest $K_n(R\Gamma)$ from the values of $\mathbb{K}R^{-\infty}(\Gamma/H)$ on the groups $H \in \mathcal{VC}$, and the structure of the restricted orbit category $\mathrm{Or}(\Gamma, \mathcal{VC})$.

The pseudo-isotopic version of the Farrell and Jones Conjecture is obtained by replacing the algebraic K-theory spectrum by the functors $\mathcal{P}_*$, $\mathcal{P}_*^{\mathrm{diff}}$, which map from the category of topological spaces X to the category of Ω-$\mathcal{SPECTRA}$. The functor $\mathcal{P}_*(?)$ (or $\mathcal{P}_*^{\mathrm{diff}}(?)$) maps the space X to the Ω-spectrum of stable topological (or smooth) pseudo-isotopies of X (see [16, Section 1.1]).

The relation between $\mathcal{P}_*(?)$ and lower algebraic K-theory is given by the work of Anderson and Hsiang [1, Theorem 3]. They show

$$\pi_j(\mathcal{P}_*(X)) = \begin{cases} Wh(\mathbb{Z}\pi_1(X)), & j = -1, \\ \tilde{K}_0(\mathbb{Z}\pi_1(X)), & j = -2, \\ K_{j+2}(\mathbb{Z}\pi_1(X)), & j \leqslant -3. \end{cases}$$

We make use of this relation in the Main Theorem.

The main result in [16] is that the Isomorphism Conjecture is true for the pseudo-isotopy and smooth pseudo-isotopy functors when $\pi_1(X) = \Gamma$ is a discrete cocompact subgroup of a virtually connected Lie group, this result together with the identification given by Anderson and Hsiang of the lower homotopy groups of the pseudo-isotopy spectrum and the lower algebraic K-theory implies the following Theorem (see [16, Section 1.6.5 and Theorem 2.1]):

THEOREM 2.1 (Farrell and Jones). *Let Γ be a cocompact discrete subgroup of a virtually connected Lie group. Then the assembly map*

$$H_n^{\mathrm{Or}(\Gamma)}(E(\Gamma, \mathcal{VC}); \mathbb{K}\mathbb{Z}^{-\infty}) \longrightarrow K_n(\mathbb{Z}\Gamma)$$

is an isomorphism for $n \leqslant 1$ and a surjection for $n = 2$.

Farrell and Jones also proved Theorem 2.1 for discrete cocompact groups, acting properly discontinuously by isometries on a simply connected Riemannian manifold M with everywhere non-positive curvature [16, Proposition 2.3]. Berkove, Farrell, Pineda, and Pearson extended this result to discrete groups, acting properly discontinuously on hyperbolic n-spaces via isometries, whose orbit space has finite volume (but not necessarily compact) (see [5, Theorem A]). In particular this result is valid for Γ a hyperbolic, non-cocompact, n-simplex reflection group.

Sometimes we can use smaller families than $\mathcal{VC}$ such as the family $\mathcal{FIN}$ of finite subgroups as explained in the next result which appeared in [21, Theorem 2.3] and for $n = \infty$ in [16, Theorem A.10].

THEOREM 2.2. *Let $\mathcal{F} \subset \mathcal{F}'$ be two families of subgroups of Γ. For each $Q \in \mathcal{F}' - \mathcal{F}$, define the induced family of subgroups $\mathcal{F}_Q$ of Q as $\mathcal{F}_Q = \{G \cap Q \,|\, G \in \mathcal{F}\}$. Suppose that for all $Q \in \mathcal{F}' - \mathcal{F}$ the assembly map*

$$H_q^{\mathrm{Or}(Q, \mathcal{F}_Q))}(E(Q, \mathcal{F}_Q); \mathbb{KZ}^{-\infty}) \longrightarrow K_q(\mathbb{Z}Q)$$

is an isomorphism for all $q \leqslant n$. Then the relative assembly map

$$H_q^{\mathrm{Or}(\Gamma, \mathcal{F})}(E(\Gamma, \mathcal{F}); \mathbb{KZ}^{-\infty}) \longrightarrow H_q^{\mathrm{Or}(\Gamma, \mathcal{F}')}(E(\Gamma, \mathcal{F}'); \mathbb{KZ}^{-\infty})$$

is an isomorphism for all $q \leqslant n$.

Our intention is to use Theorem 2.1 to compute the lower algebraic K-theory of the integral group ring $\mathbb{Z}\Gamma_3$. We would like to reduce this problem even further by applying Theorem 2.2 to the case $\mathcal{F} = \mathcal{FIN}$, $\mathcal{F}' = \mathcal{VC}$ and $n < 2$. Here our first task is to determined up to isomorphism all finite subgroups and all infinite virtually cyclic subgroups of Γ_3. Once this is established, we must check the assembly map condition given in Theorem 2.2 for each infinite virtually cyclic subgroup. If we can accomplish this, then we have shown that the relative assembly map

$$H_n^{\mathrm{Or}(\Gamma_3)}(E(\Gamma_3, \mathcal{FIN}); \mathbb{KZ}^{-\infty}) \longrightarrow H_n^{\mathrm{Or}(\Gamma_3)}(E(\Gamma_3, \mathcal{VC}); \mathbb{KZ}^{-\infty})$$

is an isomorphism for $n < 2$. By combining this assembly map with the assembly map in Theorem 2.1, we have shown that the assembly map

$$H_n^{\mathrm{Or}(\Gamma_3)}(E(\Gamma_3, \mathcal{FIN}); \mathbb{KZ}^{-\infty}) \longrightarrow K_n(\mathbb{Z}\Gamma_3)$$

is an isomorphism for all $n < 2$.

Thus to compute the lower algebraic K-theory of the integral group ring $\mathbb{Z}\Gamma_3$ it suffices to compute, for $n < 2$, the homotopy groups

$$H_n^{\mathrm{Or}(\Gamma_3)}(E(\Gamma_3, \mathcal{FIN}); \mathbb{KZ}^{-\infty}).$$

These computations are feasible using Atiyah–Hirzebruch type spectral sequences described by Davis and Lück in [15, Theorem 4.7] (see proof of Theorem 4.1), and Quinn in [28, Theorem 8.7] (see proof of Main Theorem).

THEOREM 2.3. [15,28]. *There exists an Atiyah–Hirzebruch type spectral sequence*

$$E_{p,q}^2 = H_p(E(G, \mathcal{F})/G\,;\{K_q(\mathbb{Z}G_\sigma)\}) \implies H_{p+q}^{\mathrm{Or}(G,\mathcal{F})}(E(G, \mathcal{F}); \mathbb{K}\mathbb{Z}^{-\infty}).$$

3. The Groups Γ_n

Let $\mathbb{R}^{n,1}$ denote the Minkowski space, that is, an $(n+1)$-dimensional real vector space with coordinates $x = (x_1, x_2, \ldots, x_{n+1})$, equipped with the bilinear form defined by

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \cdots - x_{n+1} y_{n+1}.$$

Hyperbolic n-space $\mathbb{H}^n$ can be defined as one sheet of the hyperboloid $\langle x, x \rangle = -1$, defined by $x_{n+1} > 0$. Let $O(n, 1)$ denote the isometry group of the bilinear form and let $O^+(n, 1)$ be the subgroup of index 2 that preserves the sheets of the hyperboloid. Then $O^+(n, 1)$ is the isometry group of the Riemannian manifold $\mathbb{H}^n$.

In the conformal ball model of hyperbolic n-space, we can identify boundary at infinity of hyperbolic n-space $\partial^\infty(\mathbb{H}^n)$, with the boundary of the ball, i.e. the sphere at infinity S_∞^{n-1}.

A *hyperbolic Coxeter n-simplex* Δ^n is an n-dimensional simplex in $\mathbb{H}^n$, all of whose dihedral angles are submultiples of π or zero. We allow a simplex in $\mathbb{H}^n$ to be unbounded with ideal vertices on the sphere at infinity of $\mathbb{H}^n$. Let S be a side of a Coxeter n-simplex Δ^n in $\mathbb{H}^n$. The *reflection* of $\mathbb{H}^n$ in the side S of Δ^n is the reflection of $\mathbb{H}^n$ in the hyperplane $\langle S \rangle$ spanned by S.

DEFINITION 3.1. *A hyperbolic Coxeter n-simplex reflection group is the group generated by reflections in the sides of a Coxeter n-simplex in $\mathbb{H}^n$.*

A *hyperbolic Coxeter n-simplex reflection group* is a discrete subgroup of isometries of $\mathbb{H}^n$, with fundamental domain its defining Coxeter n-simplex Δ^n (see [11, p. 188]). Hyperbolic Coxeter n-simplex groups arise naturally in geometry as groups of symmetries of regular tessellations of $\mathbb{H}^n$. The hyperbolic Coxeter simplices were classified by Coxeter and Whidrow [12], Lannér [20], Koszul [19] and Chein [10]. For each dimension $n \geqslant 3$, there are only finitely many hyperbolic Coxeter simplices, and such simplices exist only in dimensions $n = 2, 3, \ldots, 9$.

A *Coxeter group* W is an abstract group defined by a group presentation of the form $\langle S_i \mid (S_i S_j)^{m_{ij}} \rangle$, where

(1) the indices i, j vary over some countable index set I;
(2) m_{ij} is either a positive integer or ∞ for each i, j;
(3) $m_{ij} = m_{ji}$;
(4) $m_{ii} = 1$ for each i;
(5) $m_{ij} > 1$ if $i \neq j$;
(6) if $m_{ij} = \infty$, then the relator $(S_i S_j)^{m_{ij}}$ is deleted.

Note that if $i \neq j$, then we can obtain the relator $(S_j S_i)^{m_{ji}}$ from the relators S_i^2, S_j^2, and $(S_i S_j)^{m_{ij}}$; and therefore only one of the relators $(S_i S_j)^{m_{ij}}$ and $(S_j S_i)^{m_{ji}}$ is required and the other one may be deleted.

Let $W = \langle S_i \mid (S_i S_j)^{m_{ij}} \rangle$ be a Coxeter group. The *Coxeter graph* of W is the labeled graph with vertices I and edges

$$\{(i, j) : m_{ij} > 2\}.$$

Each edge (i, j) is labeled by m_{ij}. For simplicity, the edges with $m_{ij} = 3$ are usually not labeled in a representation of a Coxeter graph.

Let $W = \langle S_i \mid (S_i S_j)^{m_{ij}} \rangle$ be a Coxeter group. For a subset $T \subset \{S_i\}_{i \in I}$ of the generating set, W_T is defined as the subgroup of W generated by T, and is called a *parabolic subgroup*. It is known that W_T is also a Coxeter group (see [34]). If T is the empty set, then W_T is defined to be the trivial group. It is well known that any finite subgroup of a Coxeter group W is conjugate to a subgroup of a parabolic group W_T for some T subset of the generating set (see [14]).

Let Γ be the group generated by the reflections in the sides on a Coxeter n-simplex Δ^n. Let $\{S_i\}$ be the set of sides of Δ^n, and for each pair of indices i, j, let $m_{ij} = \pi / \theta(S_i, S_j)$, where $\theta(S_i, S_j)$ is the dihedral angle between S_i and S_j. Then the Coxeter group $W = \langle S_i \mid (S_i S_j)^{m_{ij}} \rangle$ is isomorphic to Γ ([29, Theorem 7.1.4]), i.e., Γ is a Coxeter group.

Let Γ_n be the subgroup of $O^+(n, 1)$ that preserves the standard integer lattice $\mathbb{Z}^{n+1} \subset \mathbb{R}^{n,1}$, that is, $\Gamma_n = O^+(n, 1) \cap GL(n+1, \mathbb{Z})$.

Since Γ_n is a subgroup of the discrete group $GL(n + 1, \mathbb{Z})$, it is also a discrete group of $O^+(n, 1)$. For $n = 3, \ldots, 9$, the group Γ_n is a hyperbolic Coxeter, noncocompact, n-simplex reflection group with fundamental domain its defining Coxeter n-simplex Δ^n (see [29, p. 301]). The Coxeter graphs of these groups are listed in Figure 1.

The groups Γ_n form a nice family of discrete subgroups of isometries of hyperbolic n-space for which the Farrell and Jones Isomorphism Conjecture in lower algebraic K-theory holds. In this paper we use this result to explicitly compute the lower algebraic K-theory of the integral group ring $\mathbb{Z}\Gamma_3$.

In order to use Theorem 2.1 to compute the lower algebraic K-theory of $\mathbb{Z}\Gamma_3$, we must first classify up to isomorphism the family $\mathcal{VC}$ of all virtually
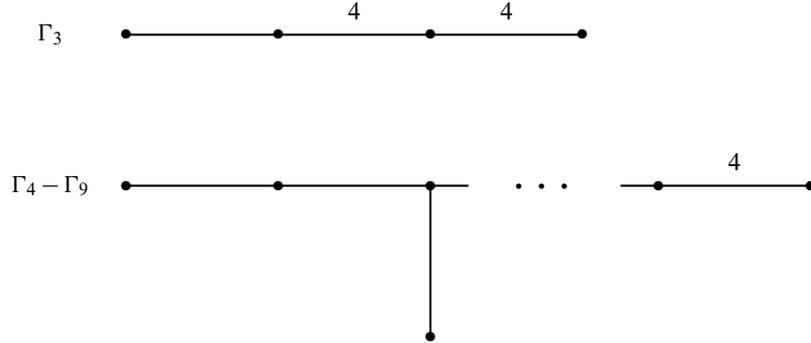
Fig. 1. The Coxeter graphs of the groups Γ_n for $n = 3, \ldots, 9$.

cyclic subgroups of Γ_3, where Γ_3 is a hyperbolic, Coxeter tetrahedra reflection group with group presentation (Figure 2)

$$\Gamma_3 = \langle S_1, S_2, S_3, S_4 \mid S_i^2 = (S_1 S_3)^2 = (S_2 S_4)^2 = (S_1 S_4)^2 = (S_1 S_2)^3$$
$$= (S_2 S_3)^4 = (S_3 S_4)^4 = 1 \rangle$$
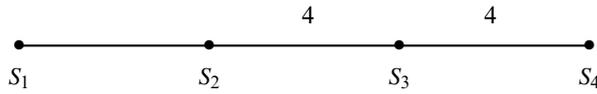
and Coxeter graph

Fig. 2. The Coxeter graph of $\Gamma_3 = [3, 4, 4]$.

Recall that a group G is *virtually cyclic* if G is either finite or contains $\mathbb{Z}$ as a subgroup of finite index. We start with the classification of the finite subgroups of Γ_3. By looking at subsets of the generating set $\{S_1, S_2, S_3, S_4\}$, the following are all parabolic subgroups of Γ_3:

$$\langle S_i \mid S_i^2 = 1 \rangle \cong \mathbb{Z}/2,$$
$$\langle S_1, S_2 \mid S_1^2 = S_2^2 = (S_1 S_2)^3 = 1 \rangle \cong D_3,$$
$$\langle S_1, S_3 \mid S_1^2 = S_3^2 = (S_1 S_3)^2 = 1 \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2,$$
$$\langle S_1, S_4 \mid S_1^2 = S_4^2 = (S_1 S_4)^2 = 1 \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2,$$
$$\langle S_2, S_3 \mid S_2^2 = S_3^2 = (S_2 S_3)^4 = 1 \rangle \cong D_4,$$
$$\langle S_2, S_4 \mid S_2^2 = S_4^2 = (S_2 S_4)^2 = 1 \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2,$$

$$\langle\, S_3,\ S_4 \mid S_3^2 = S_4^2 = (S_3 S_4)^4 = 1 \,\rangle \cong D_4,$$
$$\langle\, S_1,\ S_2,\ S_3 \mid S_1^2 = S_2^2 = S_3^2 = (S_1 S_2)^3 = (S_1 S_3)^2 = (S_2 S_3)^4 = 1 \,\rangle$$
$$\cong [3, 4] \cong \mathbb{Z}/2 \times S_4,$$
$$\langle\, S_1,\ S_2,\ S_4 \mid S_1^2 = S_2^2 = S_4^2 = (S_1 S_2)^3 = (S_1 S_4)^2 = (S_2 S_4)^2 = 1 \,\rangle$$
$$\cong \mathbb{Z}/2 \times D_3 \cong D_6,$$
$$\langle\, S_1,\ S_3,\ S_4 \mid S_1^2 = S_3^2 = S_4^2 = (S_1 S_3)^2 = (S_1 S_4)^2 = (S_3 S_4)^4 = 1 \,\rangle$$
$$\cong \mathbb{Z}/2 \times D_4,$$
$$\langle\, S_2,\ S_3,\ S_4 \mid S_2^2 = S_3^2 = S_4^2 = (S_2 S_3)^4 = (S_3 S_4)^4 = (S_2 S_4)^2 = 1 \,\rangle$$
$$\cong [4, 4] \cong P4m.$$

Here $\mathbb{Z}/n$ denotes the cyclic group of order n, D_n the dihedral group of order $2n$, S_n the symmetric group of order $n!$, and $P4m$ is a two-dimensional crystallographic group (see [28]).

Any finite subgroup of Γ_3 is conjugate to a subgroup of a parabolic group (see [14, Lemma 1.3]), thus the finite subgroups of Γ_3 up to isomorphism are: $\mathbb{Z}/2$, $\mathbb{Z}/3$, $\mathbb{Z}/4$, D_2, $\mathbb{Z}/6$, $D_3 \cong S_3$, $\mathbb{Z}/2 \times \mathbb{Z}/4$, $(\mathbb{Z}/2)^3$, D_4, D_6, A_4, $\mathbb{Z}/2 \times D_4$, S_4, $\mathbb{Z}/2 \times A_4$, $\mathbb{Z}/2 \times S_4$.

To classify the infinite virtually cyclic subgroups of Γ_3, we use the following well known result (see [17, Lemma 2.5]).

LEMMA 3.2. *Any infinite virtually cyclic group is either of type* $F \rtimes_\alpha \mathbb{Z}$, *where F is a finite group, or it maps onto D_∞ with finite kernel.*

Here $F \rtimes_\alpha \mathbb{Z}$ denotes the semi-direct product of F and $\mathbb{Z}$ where $\mathbb{Z}$ acts on F by an automorphism α of F, and D_∞ denotes the infinite dihedral group.

First, to identify the groups of type $F \rtimes_\alpha \mathbb{Z}$ which occurs in Γ_3 we observe that since F is a finite group then $F \rtimes_\alpha \mathbb{Z}$ contains $1 \times |F|! \mathbb{Z}$. Since $1 \times |F|! \mathbb{Z}$ is contained in the centralizer of F in Γ_3, then $F \rtimes_\alpha \mathbb{Z}$ can occur in Γ_3 only if $C_{\Gamma_3}(F)$ contains an element of infinite order.

There is some literature concerning centralizers of subgroups of Coxeter groups. In [6], Brink describes $C(W_T)$ in case that T consists of a single generator. Using [6, Theorem in Section 2], we can show that the centralizers $C_{\Gamma_3}(S_i)$ for $i = 1, \ldots, 4$ are the Coxeter groups given in Figure 3.

In [2], Mihalik and Bahls give a complete description of the centralizer of an arbitrary parabolic subgroup of an even Coxeter group in terms of the generators. We thank Mihalik who kindly informed us of these results and made them accessible to us. He points out that the techniques used to prove [2, Theorem 1.1] also apply to the groups Γ_n. Using this information, the following are the centralizers of the remaining parabolic subgroups of Γ_3:
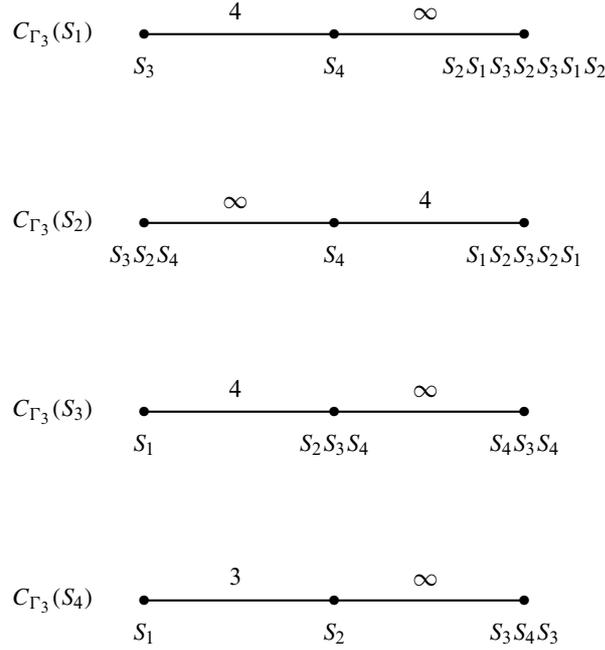
$C_{\Gamma_3}(S_1)$

$$\bullet \xrightarrow{\quad 4 \quad} \bullet \xrightarrow{\quad \infty \quad} \bullet$$

$$S_3 \qquad\qquad S_4 \qquad\qquad S_2 S_1 S_3 S_2 S_3 S_1 S_2$$

$C_{\Gamma_3}(S_2)$

$$\bullet \xrightarrow{\quad \infty \quad} \bullet \xrightarrow{\quad 4 \quad} \bullet$$

$$S_3 S_2 S_4 \qquad\qquad S_4 \qquad\qquad S_1 S_2 S_3 S_2 S_1$$

$C_{\Gamma_3}(S_3)$

$$\bullet \xrightarrow{\quad 4 \quad} \bullet \xrightarrow{\quad \infty \quad} \bullet$$

$$S_1 \qquad\qquad S_2 S_3 S_4 \qquad\qquad S_4 S_3 S_4$$

$C_{\Gamma_3}(S_4)$

$$\bullet \xrightarrow{\quad 3 \quad} \bullet \xrightarrow{\quad \infty \quad} \bullet$$

$$S_1 \qquad\qquad S_2 \qquad\qquad S_3 S_4 S_3$$

Fig. 3. The Coxeter graphs of the centralizers $C_{\Gamma_3}(S_i)$ for $i = 1, \ldots, 4$.

$$C_{\Gamma_3}(\langle S_1, S_2 \rangle) = \langle S_4 \rangle \cong \mathbb{Z}/2,$$
$$C_{\Gamma_3}(\langle S_1, S_3 \rangle) = \langle S_1, S_3, S_4 S_3 S_4 \rangle \cong (\mathbb{Z}/2)^3,$$
$$C_{\Gamma_3}(\langle S_1, S_4 \rangle) = \langle S_1, S_4, S_3 S_4 S_3 \rangle \cong (\mathbb{Z}/2)^3,$$
$$C_{\Gamma_3}(\langle S_2, S_3 \rangle) \cong \langle 1 \rangle,$$
$$C_{\Gamma_3}(\langle S_2, S_4 \rangle) = \langle S_2, S_4 \rangle \cong (\mathbb{Z}/2)^2,$$
$$C_{\Gamma_3}(\langle S_3, S_4 \rangle) = \langle S_1, (S_4 S_3)^2 \rangle \cong (\mathbb{Z}/2)^2,$$
$$C_{\Gamma_3}(\langle S_1, S_2, S_3 \rangle) \cong \langle 1 \rangle,$$
$$C_{\Gamma_3}(\langle S_1, S_2, S_4 \rangle) = \langle S_4 \rangle \cong \mathbb{Z}/2,$$
$$C_{\Gamma_3}(\langle S_1, S_3, S_4 \rangle) = \langle S_1, (S_4 S_3)^2 \rangle \cong (\mathbb{Z}/2)^2.$$

With this information we can exclude all parabolic subgroups except for those isomorphic to $\mathbb{Z}/2$.

PROPOSITION 3.3. *Let G be a subgroup of Γ_3 isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2 \cong D_2$, then $C_{\Gamma_3}(G)$ is finite.*

Proof. Let Δ^3 be the fundamental domain of Γ_3, and let V be the set of vertices of $\Delta^3 \cap \mathbb{H}^3$. Note that the fixed point set of G in $\mathbb{H}^3$ is either

$$X^G = (\mathbb{H}^3)^G = \begin{cases} *, & \text{a point in } \mathbb{H}^3, \\ L, & \text{a line in } \mathbb{H}^3, \\ P, & \text{a plane in } \mathbb{H}^3. \end{cases}$$

and $C_{\Gamma_3}(G)$ leaves invariant X^G. If $X^G = *$, then $C_{\Gamma_3}(G)$ is finite ($C_{\Gamma_3}(G)$ acts properly discontinuously on $\mathbb{H}^3$ since Γ_3 does, and fixes a point). Hence we need only to consider the case where a line $L \subset X^G$. Since $\bigcup_{\gamma \in \Gamma_3}(\Gamma_3 V)$ is countable, then $\gamma \cdot x \in \Delta^3 - V$ for some point $x \in L$ and $\gamma \in \Gamma_3$. Therefore G is conjugate to a subgroup of a parabolic subgroup of Γ_3 which is different from

$$\langle S_1, S_2, S_3 \rangle, \quad \langle S_1, S_2, S_4 \rangle, \quad \langle S_1, S_3, S_4 \rangle, \quad \langle S_2, S_3, S_4 \rangle.$$

In fact it must be either

$$P_1 = \langle S_1, S_3 \rangle, \qquad P_2 = \langle S_1, S_4 \rangle,$$

$$P_3 = \langle S_2, S_3 \rangle, \qquad P_4 = \langle S_2, S_4 \rangle \quad \text{or} \quad P_5 = \langle S_3, S_4 \rangle.$$

If $G = \gamma P_i \gamma^{-1}$ with $i = 1, 2$ or 4, then $C_{\Gamma_3}(G)$ is finite by the previous calculations. Therefore we only need to consider the two cases: $P_3 = \langle S_2, S_3 \rangle$ and $P_5 = \langle S_3, S_4 \rangle$. In both of this two cases G fixes a line L such that $\Delta^3 \cap L$ is a half line. Hence $C_{\Gamma_3}(G)$ can not contain an element of infinite order because if it does, i.e.. if there exist a $\gamma \in C_{\Gamma_3}(G)$ of infinite order, then γ leaves L invariant since $C_{\Gamma_3}(G)$ does and acts on L by isometries of L; but since $\mathrm{Isom}(L) = \mathbb{R} \rtimes \mathbb{Z}/2$, then γ acts on L by a non-trivial translation (recall that Γ_3 acts properly discontinuously on $\mathbb{H}^3$), therefore γ maps some point of $\Delta^3 \cap L$ into a different point, but this is impossible since Δ^3 is the fundamental domain for Γ_3 acting on $\mathbb{H}^3$. $\square$

The other subgroups except for $\mathbb{Z}/3$, $\mathbb{Z}/4$ and $\mathbb{Z}/6$ can not occur either, since D_2 occurs as a subgroup of each of them and $C_{\Gamma_3}(D_2)$ is finite.

Next, we study the centralizers of the remaining finite groups: $\mathbb{Z}/3$, $\mathbb{Z}/4$, and $\mathbb{Z}/6$.

As was mentioned at the beginning of this section, hyperbolic n-simplex groups arise as groups of symmetries of regular tessellations of hyperbolic n-space. The symmetric group $\Gamma_3 = [3, 4, 4]$ of a honeycomb $\{3, 4, 4\}$ is generated by reflections S_i in four planes P_i (say) which form a 3-simplex with dihedral angles

$$\theta(P_1, P_2) = \pi/3, \qquad \theta(P_2, P_3) = \pi/4, \qquad \theta(P_3, P_4) = \pi/4$$

and the remaining three angles $\pi/2$. Since the group of isometries of $\mathbb{H}^3$ is isomorphic to the group of Möbius transformations of $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and any isometry of hyperbolic 3-space is generated by reflections, then $[3, 4, 4]$

can be represented by a group of Möbius transformations generated by the inversions in four circles cutting one another at the same angles as the corresponding reflection planes. Therefore Γ_3 can be represented by the antilinear fractional transformations:

$$\Gamma_3 = [3, 4, 4] = \langle\, S_1(z) = 1/\bar{z}, \;\; S_2(z) = 1 - \bar{z}, \;\; S_3(z) = i\bar{z}, \;\; S_4(z) = \bar{z} \,\rangle.$$

Recall that an antilinear fractional transformation is a continuous map $\phi : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ of the form

$$\phi(z) = \frac{a\bar{z} + b}{c\bar{z} + d},$$

where a, b, c, d are in $\mathbb{C}$ and $ad - bc \neq 0$.

The generators of Γ_3 can be represented by antilinear fractional transformations determined by the following matrices

$$R_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \quad R_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using this representation an elementary calculation shows that the centralizer of F in Γ_3 for $F = \mathbb{Z}/3$, $\mathbb{Z}/4$, and $\mathbb{Z}/6$, contains an element of infinite order only if $F = \mathbb{Z}/3$.

Therefore we have shown that the centralizer of F in Γ_3 contains an element of infinite order only if $F = \mathbb{Z}/2$ or $\mathbb{Z}/3$.

Hence the possible infinite virtually cyclic subgroups of Γ_3 of type $F \rtimes_\alpha \mathbb{Z}$ are:

$$\mathbb{Z}, \;\; \mathbb{Z}/2 \times \mathbb{Z}, \;\; \mathbb{Z}/3 \times \mathbb{Z}, \;\; \mathbb{Z}/3 \rtimes \mathbb{Z}.$$

Next, we classify the groups that map onto D_∞ with finite kernel.

Let Q map onto D_∞, with non-trivial kernel, i.e., we are given the following exact sequence

$$1 \to F \to Q \to D_\infty \to 1.$$

Now, this exact sequence gives rise to another exact sequence

$$1 \to F \to Q' \to \mathbb{Z} \to 1,$$

where $\mathbb{Z}$ is the infinite cyclic subgroup of index two in D_∞ and Q' is just the inverse of $\mathbb{Z}$ under the map $Q \to D_\infty$. The sequence $1 \to F \to Q' \to Z \to 1$ splits since $\mathbb{Z}$ is free. Hence F is isomorphic to either $\mathbb{Z}/2$ or $\mathbb{Z}/3$ and Q' is isomorphic to either $\mathbb{Z}/2 \times \mathbb{Z}$, $\mathbb{Z}/3 \times \mathbb{Z}$, or $\mathbb{Z}/3 \rtimes \mathbb{Z}$ by the classification of the groups of type $F \rtimes_\alpha \mathbb{Z}$. Since Q' is a subgroup of index two in Q, we have the exact sequence

$$1 \to Q' \to Q \to \mathbb{Z}/2 \to 1.$$

Thus we have reduced the classification of the groups that map onto D_∞ to finding solutions to the extension problems given above.

Using group cohomology (see [7, Theorems IV.3.12, and IV.6.6]) a fairly straight-forward calculation shows that Q must be one of the following groups (similar calculations also appear for example in [5, 21, 33]):

$$D_\infty, \ \mathbb{Z}/2 \times D_\infty, \ \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/4, \ \mathbb{Z}/4 *_{\mathbb{Z}/2} (\mathbb{Z}/2)^2,$$

$$\mathbb{Z}/3 \times D_\infty \cong \mathbb{Z}/6 *_{\mathbb{Z}/3} \mathbb{Z}/6, \ S_3 *_{\mathbb{Z}/3} S_3.$$

Hence we have a proof of the following lemma:

LEMMA 3.4. *Let Q be infinite virtually cyclic subgroup of Γ_3, then Q is one of the following groups:*

$$\mathbb{Z}, \ \mathbb{Z}/2 \times \mathbb{Z}, \ \mathbb{Z}/3 \times \mathbb{Z}, \ \mathbb{Z}/3 \rtimes \mathbb{Z}, \ D_\infty, \ \mathbb{Z}/2 \times D_\infty, \ \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/4,$$

$$\mathbb{Z}/4 *_{\mathbb{Z}/2} (\mathbb{Z}/2)^2, \ \mathbb{Z}/3 \times D_\infty \cong \mathbb{Z}/6 *_{\mathbb{Z}/3} \mathbb{Z}/6, \ S_3 *_{\mathbb{Z}/3} S_3.$$

4. The Reduction to Finite Subgroups

In Theorem 4.7, we show that for $\Gamma = \Gamma_3$, $\mathcal{F} = \mathcal{FIN}$ and $\mathcal{F}' = \mathcal{VC}$ the relative assembly map given in Theorem 2.2 is an isomorphism for $n < 2$. Thus it suffices to compute $H_n^{\mathrm{Or}(\Gamma_3)}(E(\Gamma_3, \mathcal{F}); \mathbb{K}R^{-\infty})$ when $\mathcal{F} = \mathcal{FIN}$ to get the main result.

THEOREM 4.1. *The assembly map*

$$\mathcal{A}_{\mathcal{F}_Q} : H_n^{\mathrm{Or}(Q, \mathcal{F}_Q)}(E(Q, \mathcal{F}_Q); \mathbb{K}\mathbb{Z}^{-\infty}) \longrightarrow K_n(\mathbb{Z}Q)$$

is an isomorphism for any $n \in \mathbb{Z}$ if $Q = \mathbb{Z}$ or D_∞, and an isomorphism for $n < 2$ if $Q = \mathbb{Z}/2 \times \mathbb{Z}, \ \mathbb{Z}/3 \times \mathbb{Z}, \ \mathbb{Z}/3 \rtimes \mathbb{Z}, \ \mathbb{Z}/2 \times D_\infty, \ \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/4, \ \mathbb{Z}/4 *_{\mathbb{Z}/2} (\mathbb{Z}/2)^2, \ \mathbb{Z}/6 *_{\mathbb{Z}/3} \mathbb{Z}/6, \ or \ S_3 *_{\mathbb{Z}/3} S_3.$

Note that the family $\mathcal{F}_Q$ consists of the finite subgroups of Q.

Remark 4.2. The claim for $\mathbb{Z}$ and D_∞ has been proved in [21, Lemma 2.4] for any n.

To prove Theorem 4.1 for the remaining subgroups we need a sequence of lemmas. In Lemma 4.3, we compute the K-theory of each possible Q, the range of $\mathcal{A}_{\mathcal{F}_Q}$, and we observe in Proposition 4.5 that the assembly map

$$\mathcal{A} : H_n^{\mathrm{Or}(Q, \mathcal{TR})}(E(Q, \mathcal{TR}); \mathbb{K}\mathbb{Z}^{-\infty}) \longrightarrow K_n(\mathbb{Z}Q)$$

is an isomorphism for $n < 2$ in each case except for $Q = \mathbb{Z}/6 *_{\mathbb{Z}/3} \mathbb{Z}/6$ for which $\mathcal{A}$ fails to be an isomorphism at $n = -1$ (see Remark 4.6). Then to complete the proof of the theorem, we compute $H_n^{\mathrm{Or}(Q)}(E(Q, \mathcal{F}_Q); \mathbb{K}Z^{-\infty})$, the domain of $\mathcal{A}_{\mathcal{F}_Q}$, and note that the domain and the range of $A_{\mathcal{F}_Q}$ are finitely generated abelian groups, and are isomorphic as abstracts groups for $n < 2$. We use the fact that the assembly map $\mathcal{A}$ factors through $\mathcal{A}_{\mathcal{F}_Q}$; this guarantees that $\mathcal{A}_{\mathcal{F}_Q}$ is surjective. For $Q = \mathbb{Z}/6 *_{\mathbb{Z}/3} \mathbb{Z}/6$ [17, Theorem 2.6] guarantees the surjectivity of $\mathcal{A}_{\mathcal{F}_Q}$ at $n = -1$. For finitely generated abelian groups, any surjective endomorphism is automatically an isomorphism completing the proof.

LEMMA 4.3. *Let Q be an infinite virtually cyclic subgroup of Γ_3. If $Q \neq \mathbb{Z}/6 *_{\mathbb{Z}/3} \mathbb{Z}/6$, then $Wh(Q) = 0$, $K_0(\mathbb{Z}Q) = \mathbb{Z}$, and $K_n(\mathbb{Z}Q) = 0$ for all $n < 0$. If $Q = \mathbb{Z}/6 *_{\mathbb{Z}/3} \mathbb{Z}/6$, then $Wh(Q) = 0$, $K_0(\mathbb{Z}Q) = \mathbb{Z}$, $K_{-1}(\mathbb{Z}Q) = \mathbb{Z} \oplus \mathbb{Z}$, and $K_n(\mathbb{Z}Q) = 0$ for all $n \leqslant -2$.*

In order to carry out our computations, we need information on the K-theory of the finite subgroups of each infinite virtually cyclic group. These finite subgroups are: $\mathbb{Z}/2$, $\mathbb{Z}/3$, $\mathbb{Z}/4$, $\mathbb{Z}/2 \times \mathbb{Z}/2 \cong D_2$, $\mathbb{Z}/6$, $S_3 \cong D_3$.

It is a result of Carter [8] that $K_n(\mathbb{Z}G) = 0$ if $n < -1$ for any finite group G. In [4, Theorem 10.6], Bass determines $K_{-1}(\mathbb{Z}G)$ for finite abelian groups. For $G = \mathbb{Z}/2$, $\mathbb{Z}/3$, $\mathbb{Z}/4$ and $(\mathbb{Z}/2)^2$, $K_{-1}(\mathbb{Z}G) = 0$. For $G = \mathbb{Z}/6$, $K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) = \mathbb{Z}$. To compute $K_{-1}(\mathbb{Z}[D_3])$ we need the following formula due to Carter [9, Theorem 3], the reader is referred to Section 5.

$$0 \to K_0(\mathbb{Z}) \to \left(\bigoplus_{p|n} K_0(\hat{\mathbb{Z}}_p D_3) \right) \oplus K_0(\mathbb{Q}D_3) \to \bigoplus_{p|n} K_0(\hat{\mathbb{Q}}_p D_3) \to K_{-1}(\mathbb{Z}D_3) \to 0.$$

The group algebra $\mathbb{Q}D_3$ is isomorphic to $\mathbb{Q} \times \mathbb{Q} \times M_2(\mathbb{Q})$, and the same statement is true if $\mathbb{Q}$ is replaced by $\hat{\mathbb{Q}}_2$ or $\hat{\mathbb{Q}}_3$. Hence $K_0(\hat{\mathbb{Q}}_2[D_3]) \cong K_0(\hat{\mathbb{Q}}_3[D_3]) \cong K_0(\mathbb{Q}[D_3]) \cong \mathbb{Z}^3$. Using techniques described in [13, Section 5], we have that $K_0(\hat{\mathbb{Z}}_3[D_3]) \cong K_0(\mathbb{F}_3[D_3]) \cong K_0(\mathbb{F}_3[\mathbb{Z}/2]) = K_0(\mathbb{F}_3 \times \mathbb{F}_3) = \mathbb{Z}^2$. Also $K_0(\hat{\mathbb{Z}}_2[D_3]) \cong K_0(\mathbb{F}_2[D_3]) = K_0(\mathbb{F}_2[\mathbb{Z}/2] \times M_2(\mathbb{F}_2)) = K_0(\mathbb{F}_2 \times M_2(\mathbb{F}_2)) = \mathbb{Z}^2$. Another result of Carter [8, Theorem 1] states that $K_{-1}(\mathbb{Z}G)$ is torsion-free for any symmetric group S_n. In particular $K_{-1}(\mathbb{Z}D_3)$ is torsion-free, so by counting ranks in the exact sequence above we have that $K_{-1}(\mathbb{Z}D_3) = 0$ (see also [26, p. 273]).

It is well known that $\tilde{K}_0(\mathbb{Z}G) = 0$ when G is any of the above finite groups (see [30, 31]), and that $Wh(G) = 0$ for G cyclic of order 2, 3, 4 and 6 (see [25]). For $Wh(D_n)$ we use the following formula given in [3]: $Wh(G) = \mathbb{Z}^y \oplus SK_1(\mathbb{Z}G)$, where y is the number of irreducible real representations of G minus the number of irreducible rational representations of G. When $G = D_2$ or D_3, y is zero, and Oliver [25] proves $SK_1(\mathbb{Z}G)$ vanishes for all

finite dihedral groups. For each of the finite groups in question, $K_1(\mathbb{Z}G) = G^{ab} \oplus \mathbb{Z}/2$ (see [25]).

We are now ready to prove Lemma 4.3.

Proof of Lemma 4.3. If $Q = \mathbb{Z}/p \times \mathbb{Z}$ with $p = 2$ or 3, we write $\mathbb{Z}[\mathbb{Z}/p][\mathbb{Z}]$, and apply the Fundamental Theorem of algebraic K-theory (see [4, Theorem 10.6]):

$$K_n(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}]) \cong K_n(\mathbb{Z}[\mathbb{Z}/p]) \oplus K_{n-1}(\mathbb{Z}[\mathbb{Z}/p]) \oplus 2NK_n(\mathbb{Z}[\mathbb{Z}/p]), \quad n < 2.$$

In [21, Theorem 3.1 (e)] Lück and Stamm show that the nil terms $NK_n(\mathbb{Z}[\mathbb{Z}/p])$ are zero for $n < 2$. Thus for $p = 2$ or 3

$$K_n(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}]) \cong K_n(\mathbb{Z}[\mathbb{Z}/p]) \oplus K_{n-1}(\mathbb{Z}[\mathbb{Z}/p]) = 0, \quad n \leqslant -1,$$
$$K_0(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}]) \cong K_0(\mathbb{Z}[\mathbb{Z}/p]) \oplus K_{-1}(\mathbb{Z}[\mathbb{Z}/p]) \cong \mathbb{Z},$$
$$K_1(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}]) \cong K_1(\mathbb{Z}[\mathbb{Z}/p]) \oplus K_0(\mathbb{Z}[\mathbb{Z}/p]) \cong \mathbb{Z}/p \oplus \mathbb{Z}/2 \oplus \mathbb{Z}.$$

Hence it follows that both $Wh(Q)$ and $\tilde{K}_0(Q)$ vanish for $Q = \mathbb{Z}/2 \times \mathbb{Z}$ or $\mathbb{Z}/3 \times \mathbb{Z}$, (see also [5, Proposition 15]).

The lower algebraic K-theory of $Q = \mathbb{Z}/3 \rtimes \mathbb{Z}$ has already been computed in [5, Proposition 15]. They show: $K_1(\mathbb{Z}[\mathbb{Z}/3 \rtimes \mathbb{Z}]) = \mathbb{Z} \oplus \mathbb{Z}/2$, $Wh(\mathbb{Z}[\mathbb{Z}/3 \rtimes \mathbb{Z}]) = 0$, $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/3 \rtimes \mathbb{Z}]) = 0$, and $K_n(\mathbb{Z}[\mathbb{Z}/3 \rtimes \mathbb{Z}]) = 0$ for $n \leqslant -1$.

The other cases are groups of the shape $Q = Q_0 *_{\mathbb{Z}/p} Q_1$, with $Q \neq \mathbb{Z}/6 *_{\mathbb{Z}/3} \mathbb{Z}/6$, and finite groups Q_i such that $Wh_q(Q_i) = 0$ for $q \leqslant 1$ (to clarify this notation we refer the reader to Section 5) . In [17] Farrell and Jones show that if G is infinite virtually cyclic, then $K_n(\mathbb{Z}G)$ is zero for $n < -1$ and that $K_{-1}(\mathbb{Z}G)$ is generated by the images of $K_{-1}(\mathbb{Z}F)$ where F ranges over all finite subgroups $F \subset G$. Then it follows that $K_n(\mathbb{Z}Q) = 0$ for all $n < 0$.

Prassidis and Munkholm in [24, Corollary 3.6] show that there are exact sequences

$$K_1(\mathbb{Z}[\mathbb{Z}/p]) \to K_1(\mathbb{Z}Q_0) \oplus K_1(\mathbb{Z}Q_1) \to K_1(\mathbb{Z}Q) \to K_0(\mathbb{Z}[\mathbb{Z}/p]) \to \cdots,$$

and

$$Wh(\mathbb{Z}/p) \to Wh(Q_0) \oplus Wh(Q_1) \to Wh(Q) \to \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/p]) \to \cdots$$

After working through the exact sequences, we see that for all four groups Q, $K_0(\mathbb{Z}Q) = \mathbb{Z}$, $K_1(\mathbb{Z}Q) = Q^{ab} \oplus \mathbb{Z}/2$, and $Wh(Q) = 0$. The result for $Q = \mathbb{Z}/2 \times D_\infty$, and $Q = S_3 *_{\mathbb{Z}/3} S_3$ also appears in [5, Proposition 15].

The claim for $Q = \mathbb{Z}/6 *_{\mathbb{Z}/3} \mathbb{Z}/6$ follows from the arguments used in the last two paragraphs: $K_{-1}(\mathbb{Z}Q) = \mathbb{Z} \oplus \mathbb{Z}$ and $K_n(\mathbb{Z}Q) = 0$ for $n \leqslant -2$ by [17]. $K_0(\mathbb{Z}Q) = \mathbb{Z}$ and $Wh(Q) = 0$ by [24]. $\qquad\square$

We will make repeated use of the following well known lemma for an arbitrary group G (see [26, Lemma 1.1]):

LEMMA 4.4. *For $n < 0$, the assembly map $H_n(BG; \mathbb{K}^{-\infty}(\mathbb{Z})) \longrightarrow K_n(\mathbb{Z}G)$ is an isomorphism if and only if $K_n(\mathbb{Z}G) = 0$. It is an isomorphism for $n = 0$ if and only if $\tilde{K}_0(\mathbb{Z}G) = 0$, and it is an isomorphism for $n = 1$ if and only if $Wh(G) = 0$.*

PROPOSITION 4.5. *Let Q be an infinite virtually cyclic subgroup of Γ_3. If $Q \neq \mathbb{Z}/6 *_{\mathbb{Z}/3} \mathbb{Z}/6$, then the assembly map*

$$\mathcal{A} : H_n^{\mathrm{Or}(Q, \mathcal{TR})}(E(Q, \mathcal{TR}); \mathbb{K}\mathbb{Z}^{-\infty}) = H_n(BQ; \mathbb{K}^{\infty}(\mathbb{Z})) \longrightarrow K_n(\mathbb{Z}Q)$$

is an isomorphism for $n < 2$. If $Q = \mathbb{Z}/6 *_{\mathbb{Z}/3} \mathbb{Z}/6$, then $\mathcal{A}$ is an isomorphism for $n = 0, 1$, and $n \leqslant -2$.

Proof. This follows immediately from Lemmas 4.3 and 4.4. $\qquad\square$

Remark 4.6. If $Q = \mathbb{Z}/6 *_{\mathbb{Z}/3} \mathbb{Z}/6$, then the assembly map $\mathcal{A}$ can not be an isomorphism for $n = -1$, since $K_{-1}(\mathbb{Z}Q) = \mathbb{Z} \oplus \mathbb{Z}$.

Now we are ready to complete the proof of Theorem 4.1.

Proof. [Proof of Theorem 4.1.] In Lemma 4.3, we compute the target of the map for each possible Q. We now compute the domain using the Atiyah–Hirzerbruch type spectral sequence given in Theorem 2.3 (see [15, Theorem 4.7]).

$$E_{p,q}^2 = H_p(E(Q, \mathcal{F})/Q; \{K_q(\mathbb{Z}Q_\sigma)\}) \Longrightarrow H_{p+q}^{\mathrm{Or}(Q, \mathcal{F})}(E(Q, \mathcal{F}); \mathbb{K}\mathbb{Z}^{-\infty})$$

with $\mathcal{F} = \mathcal{F}_Q$. Note that the family $\mathcal{F}_Q$ consists of the finite subgroups of Q.

The infinite cyclic group acts on the real line (with an appropriate simplicial decomposition) in an obvious way with finite isotropy, and this action can be extended to the groups with $\mathbb{Z}/2$ and $\mathbb{Z}/3$ summands by letting $\mathbb{Z}/2$ and $\mathbb{Z}/3$ act trivially. Hence for the groups of type $F \rtimes \mathbb{Z}$, $E(Q, \mathcal{F}_Q) = \mathbb{R}$, and $E(Q, \mathcal{F}_Q)/Q = S^1$ with $\mathbb{Z}/2$, and $\mathbb{Z}/3$ stabilizers at each simplex, respectively.

For the groups of the shape $Q = Q_0 *_{\mathbb{Z}/p} Q_1$ with $p = 2$ or 3, $E(Q, \mathcal{F}_Q)/Q$ is an interval, with $\mathbb{Z}/p$ stabilizer at the 1-simplex, and Q_i stabilizers at each vertex (see [32, Theorem 7]).

The rings $\mathbb{Z}Q_\sigma$ for each of the stabilizers all have trivial negative K-groups, (except for $Q_\sigma = \mathbb{Z}/6$ for which $K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) = \mathbb{Z}$), $K_0(\mathbb{Z}Q_\sigma) \cong \mathbb{Z}$ and $K_1(\mathbb{Z}Q_\sigma) \cong Q_\sigma^{ab} \oplus \mathbb{Z}/2$.

With the above information, the E^2 term of the homotopy colimit spectral sequence can be easily computed for n less than two. Since

$E(Q, \mathcal{F}_Q)/Q$ is one dimensional, the spectral sequence collapses at E^2, and in each case

$$H_n^{\mathrm{Or}(Q,\mathcal{F}_Q)}(E(Q, \mathcal{F}_Q); \mathbb{KZ}^{-\infty}) \cong K_n(\mathbb{Z}Q) \quad \text{for} \quad n < 2.$$

It remains to show that $\mathcal{A}_{\mathcal{F}_Q}$ gives the isomorphism.

Recall that the assembly map $\mathcal{A}$ factors through the assembly map:

$$H_n(BQ; \mathbb{K}^{-\infty}(\mathbb{Z})) \longrightarrow H_n^{\mathrm{Or}(Q,\mathcal{F}_Q)}(E(Q, \mathcal{F}_Q); \mathbb{KZ}^{-\infty}) \longrightarrow K_n(\mathbb{Z}Q).$$

To finish the argument, we consider the following two cases:

(1) If $Q \neq \mathbb{Z}/6 *_{\mathbb{Z}/3} \mathbb{Z}/6$, then Proposition 4.5 implies that the assembly map is an isomorphism for $n < 2$, thus the composite is an isomorphism. This guarantees the assembly map $\mathcal{A}_{\mathcal{F}_Q}$ is surjective in this case.
(2) If $Q = \mathbb{Z}/6 *_{\mathbb{Z}/3} \mathbb{Z}/6$, then Proposition 4.5 implies that the assembly map $\mathcal{A}_{\mathcal{F}_Q}$ is surjective for all $n < 2$, with $n \neq -1$ (see Remark 4.6) [17, Theorem 2.6] guarantees that $\mathcal{A}_{\mathcal{F}_Q}$ is surjective at $n = -1$.

Since $K_0(\mathbb{Z}Q)$, $K_1(\mathbb{Z}Q)$, and $K_{-1}(\mathbb{Z}[\mathbb{Z}/6 *_{\mathbb{Z}/3} \mathbb{Z}/6])$ (for the other infinite virtually cyclic groups $K_{-1}(\mathbb{Z}Q) = 0$) are finitely generated abelian groups, any surjective endomorphism is an automorphism completing the proof of Theorem 4.1. $\square$

THEOREM 4.7. *The relative assembly map*

$$H_n^{\mathrm{Or}(\Gamma_3)}(E(\Gamma_3, \mathcal{FIN}); \mathbb{KZ}^{-\infty}) \longrightarrow H_n^{\mathrm{Or}(\Gamma_3)}(E(\Gamma_3, \mathcal{VC}); \mathbb{KZ}^{-\infty})$$

is an isomorphism for $n < 2$.

Proof. It follows from Theorems 4.1 and 2.2. $\square$

COROLLARY 4.8. *The assembly map*

$$H_n^{\mathrm{Or}(\Gamma_3)}(E(\Gamma_3, \mathcal{FIN}); \mathbb{KZ}^{-\infty}) \longrightarrow K_n(\mathbb{Z}\Gamma_3)$$

is an isomorphism for all $n < 2$.

Proof. It follows from Theorems 4.7 and 2.1. $\square$

5. Proof of Main Theorem

MAIN THEOREM 2. *Let* $\Gamma_3 = O^+(3,1) \cap GL(4,\mathbb{Z})$. *Then the lower algebraic K-theory of the integral group ring of Γ_3 is given as follows:*

$$Wh(\Gamma_3) = 0,$$
$$\tilde{K}_0(\mathbb{Z}\Gamma_3) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/4,$$
$$K_{-1}(\mathbb{Z}\Gamma_3) \cong \mathbb{Z} \oplus \mathbb{Z},$$
$$K_n(\mathbb{Z}\Gamma_3) = 0 \quad \text{for } n < -1.$$

Proof. We use Quinn's spectral sequence for the pseudo-isotopy spectrum $\mathcal{P}_*$ (see Section 2 [16, Lemma 1.4.2], and [28, Theorem 8.7]),

$$E^2_{p,q} = H_p(E(G,\mathcal{F})/G\,;\{Wh_q(G_\sigma)\}) \implies Wh_{p+q}(G).$$

All the information needed to compute the E^2-term is encoded in $E(\Gamma_3, \mathcal{FIN})/\Gamma_3$ and in the algebraic K-groups of the finite subgroups of Γ_3.

Since Γ_3 is a hyperbolic, Coxeter tetrahedra reflection group, the fundamental domain Δ^3 of Γ_3 satisfies the requirements to be a model for $E(\Gamma_3, \mathcal{FIN})/\Gamma_3$. Δ^3 has 4 faces with stabilizer $\mathbb{Z}/2$, 6 edges with stabilizers D_2, D_3, D_4, three vertices with stabilizers D_6, $\mathbb{Z}/2 \times D_4$, $\mathbb{Z}/2 \times S_4$, and one ideal vertex with stabilizer the crystallographic group $P4m$ (in our calculations the ideal vertex is being ignored); in [26] Pearson shows that the lower algebraic K-theory of $P4m$ vanishes.

The complex that gives the homology of Δ^3 with local coefficients $\{Wh_q(F_\sigma)\}$ has the form

$$\bigoplus_{\sigma^2} Wh_q(F_{\sigma^2}) \to \bigoplus_{\sigma^1} Wh_q(F_{\sigma^1}) \to \bigoplus_{\sigma^0} Wh_q(F_{\sigma^0}),$$

where σ^i denotes the cells in dimension i, and $Wh_q(F_{\sigma^i})$ occurs in the summand as many times as the numbers of conjugacy classes of the subgroup F_{σ^i} in Γ_3. The homology of this complex gives the data for the E^2-term. Let us recall that

$$Wh_q(F) = \begin{cases} Wh(F), & q = 1, \\ \tilde{K}_0(\mathbb{Z}F), & q = 0, \\ K_n(\mathbb{Z}F), & q \leqslant -1. \end{cases}$$

So we analyze this complex for each of the following cases: $q < -1$, $q = -1, 0, 1$.

$q < -1$. Carter shows in [8] that $K_q(\mathbb{Z}F) = 0$ when F is a finite group. Hence the whole complex consists of zero terms and we obtain $E^2_{p,q} = 0$ for $q < -1$.

$q = -1$. Again using Carter's result in [8], $K_{-1}(\mathbb{Z}F) = 0$, for all the groups which occur as stabilizers of the 2-cells and the 1-cells, therefore $E^2_{p,-1} = 0$ for $p = 1, 2$. For $p = 0$ the complex may have non-zero terms in dimension zero, and the resulting homology group is

$$H_0(\Delta^3; \{K_{-1}(\mathbb{Z}F_\sigma)\}) = \bigoplus_{\sigma^0} K_{-1}(\mathbb{Z}F_{\sigma^0}).$$

Since there is only one conjugacy class for each of the subgroups D_6, $\mathbb{Z}/2 \times D_4$, $\mathbb{Z}/2 \times S_4$ of Γ_3 occurring as stabilizers of the 0-cells, then

$$H_0(\Delta^3; \{K_{-1}(\mathbb{Z}F_\sigma)\} = K_{-1}(\mathbb{Z}[D_6]) \oplus K_{-1}(\mathbb{Z}[\mathbb{Z}/2 \times D_4])$$
$$\oplus K_{-1}(\mathbb{Z}[\mathbb{Z}/2 \times S_4]).$$

To calculate the K-groups: $K_{-1}(\mathbb{Z}[D_6])$, $K_{-1}(\mathbb{Z}[\mathbb{Z}/2 \times D_4])$, and $K_{-1}(\mathbb{Z}[\mathbb{Z}/2 \times S_4])$, we use the following formula due to Carter [9, Theorem 3].

Let G be a group of order n, let p denote a prime number, let $\hat{\mathbb{Z}}_p$ denote the p-adic integers and let $\hat{\mathbb{Q}}_p$ denote the p-adic numbers. Then the following sequence is exact:

$$0 \to K_0(\mathbb{Z}) \to \left(\bigoplus_{p|n} K_0(\hat{\mathbb{Z}}_p G)\right) \oplus K_0(\mathbb{Q}G) \to \bigoplus_{p|n} K_0(\hat{\mathbb{Q}}_p G) \to K_{-1}(\mathbb{Z}G) \to 0.$$

The group algebra $\mathbb{Q}D_6$ is isomorphic to $\mathbb{Q}^4 \times M_2(\mathbb{Q}) \times M_2(\mathbb{Q})$, and the same statement is true if $\mathbb{Q}$ is replaced by $\hat{\mathbb{Q}}_2$ and $\hat{\mathbb{Q}}_3$. Hence $K_0(\mathbb{Q}D_6) \cong K_0(\hat{\mathbb{Q}}_2[D_6]) \cong K_0(\hat{\mathbb{Q}}_3[D_6]) \cong \mathbb{Z}^6$. Using techniques described in [13, Section 5], $K_0(\hat{\mathbb{Z}}_2[D_6]) \cong K_0(\mathbb{F}_2[D_6]) \cong K_0(\mathbb{F}_2[D_3]) \cong K_0(\mathbb{F}_2[\mathbb{Z}/2] \times M_2(\mathbb{F}_2)) \cong K_0(\mathbb{F}_2 \times \mathbb{F}_2) \cong \mathbb{Z}^2$. Also $K_0(\hat{\mathbb{Z}}_3[D_6]) \cong K_0(\mathbb{F}_3[D_6]) \cong K_0(\mathbb{F}_3[D_2]) \cong K_0(\mathbb{F}_3 \times \mathbb{F}_3 \times \mathbb{F}_3 \times \mathbb{F}_3) \cong \mathbb{Z}^4$. Carter also shows in [8] that $K_{-1}(\mathbb{Z}G)$ is torsion free for any of the groups above, so counting ranks in the exact sequence, we have that $K_{-1}(\mathbb{Z}[D_6]) \cong \mathbb{Z}$ (see also [26, p. 274]).

The computations for $\mathbb{Z}/2 \times D_4$, and $\mathbb{Z}/2 \times S_4$ are nearly identical. The algebra $\mathbb{Q}[\mathbb{Z}/2 \times D_4]$ is isomorphic to $\mathbb{Q}^8 \times M_2(\mathbb{Q}) \times M_2(\mathbb{Q})$ and $\hat{\mathbb{Q}}_2[\mathbb{Z}/2 \times D_4]$ has the same decomposition, so $K_0(\mathbb{Q}[\mathbb{Z}/2 \times D_4]) \cong K_0(\hat{\mathbb{Q}}_2[\mathbb{Z}/2 \times D_4]) \cong \mathbb{Z}^{10}$. We have that $K_0(\hat{\mathbb{Z}}_2[\mathbb{Z}/2 \times D_4]) \cong K_0(\mathbb{F}_2[\mathbb{Z}/2 \times D_4]) \cong K_0(\mathbb{F}_2[D_4]) \cong K_0(\mathbb{F}_2[(\mathbb{Z}/2)^2]) \cong K_0(\mathbb{F}_2) \cong \mathbb{Z}$. Counting ranks, $K_{-1}(\mathbb{Z}[\mathbb{Z}/2 \times D_4]) \cong 0$. The algebra $\mathbb{Q}[\mathbb{Z}/2 \times S_4]$ is isomorphic to $\mathbb{Q}^4 \times (M_2(\mathbb{Q}))^2 \times (M_3(\mathbb{Q}))^4$ and the same statement is true if $\mathbb{Q}$ is replaced by $\hat{\mathbb{Q}}_2$ and $\hat{\mathbb{Q}}_3$. We have $K_0(\mathbb{Q}[\mathbb{Z}/2 \times S_4]) \cong K_0(\hat{\mathbb{Q}}_2[\mathbb{Z}/2 \times S_4]) \cong K_0(\hat{\mathbb{Q}}_3[\mathbb{Z}/2 \times S_4]) \cong \mathbb{Z}^{10}$. The integral p-adic terms are $K_0(\hat{\mathbb{Z}}_2[\mathbb{Z}/2 \times S_4]) \cong K_0(\mathbb{F}_2[\mathbb{Z}/2 \times S_4]) \cong K_0(\mathbb{F}_2[S_4]) \cong \mathbb{Z}^2$, and $K_0(\hat{\mathbb{Z}}_3[\mathbb{Z}/2 \times S_4]) \cong K_0(\mathbb{F}_3[\mathbb{Z}/2 \times S_4]) \cong K_0(\mathbb{F}_3[(\mathbb{Z}/2)^4]) \cong \mathbb{Z}^8$. Counting ranks, $K_{-1}(\mathbb{Z}[\mathbb{Z}/2 \times S_4]) \cong \mathbb{Z}$.

It follows that

$$E^2_{0,-1} = H_0(\Delta^3; \{K_{-1}(\mathbb{Z}F_\sigma)\}) = \mathbb{Z} \oplus \mathbb{Z}.$$

$q = 0$. It is well know that $\tilde{K}_0(\mathbb{Z}F) = 0$ when F is one of the groups that occur as stabilizers of the 2-cells and the 1-cells (see e.g. [30, 31]), so $E^2_{p,0} = 0$ for $p = 1, 2$. For $p = 0$ the complex may have non-zero terms in dimension zero, and the resulting homology is

$$H_0(\Delta^3; \{\tilde{K}_0(\mathbb{Z}F_\sigma)\}) = \tilde{K}_0(\mathbb{Z}[D_6]) \oplus \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/2 \times D_4]) \oplus \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/2 \times S_4]).$$

In [30] Reiner shows that $\tilde{K}_0(\mathbb{Z}[D_6]) = 0$, then

$$E^2_{0,0} = \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/2 \times D_4]) \oplus \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/2 \times S_4]).$$

To compute the K-groups: $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/2 \times G])$ for $G = D_4$ or S_4, consider the following Cartesian square

$$
\begin{array}{ccc}
\mathbb{Z}[\mathbb{Z}/2][G] & \longrightarrow & \mathbb{Z}[G] \\
\downarrow & & \downarrow \\
\mathbb{Z}[G] & \longrightarrow & \mathbb{F}_2[G],
\end{array}
$$

which yields the Mayer-Vietories sequence (see [40, Theorem 49.27])

$$K_1(\mathbb{Z}G) \oplus K_1(\mathbb{Z}G) \xrightarrow{\varphi} K_1(\mathbb{F}_2[G]) \rightarrow$$
$$\rightarrow \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/2][G]) \rightarrow \tilde{K}_0(\mathbb{Z}G) \oplus \tilde{K}_0(\mathbb{Z}G) \rightarrow 0 \qquad (1)$$

In [30] Reiner shows that $\tilde{K}_0(\mathbb{Z}G)$ is trivial for $G = D_4$ and S_4 and the K-groups $K_1(\mathbb{Z}G)$ can be computed as follows: for any group G, it is well known that the rank of $K_1(\mathbb{Z}G)$ is equal to the rank of $Wh(G)$ (see e.g. [25]) where the rank of $Wh(G)$ is the number of irreducible real representations of G minus the number of irreducible rational representations of G (see case $q = 1$ below), and that the torsion part of $K_1(\mathbb{Z}G)$ is $\{\pm 1\} \oplus G^{ab} \oplus SK_1(\mathbb{Z}G)$ (see [25, Theorem 7.4]). For $G = D_4$ or S_4, Oliver in [25] shows that both $Wh(G)$ and $SK_1(\mathbb{Z}G)$ are zero. Since $(D_4)^{ab} = \mathbb{Z}/2 \oplus \mathbb{Z}/2$, and $(S_4)^{ab} = \mathbb{Z}/2$, then it follows that $K_1(\mathbb{Z}[D_4]) = (\mathbb{Z}/2)^3$, and $K_1(\mathbb{Z}[S_4]) = (\mathbb{Z}/2)^2$. We thank B. Magurn who kindly inform us that $K_1(\mathbb{F}_2[D_4]) = \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^2$ (see [18, Lemma 4.2]), and $K_1(\mathbb{F}_2[S_4]) = \mathbb{Z}/4 \oplus \mathbb{Z}/2$ (see [22]). Therefore the exact sequence in (1) yields the following exact sequences

$$(\mathbb{Z}/2)^3 \oplus (\mathbb{Z}/2)^3 \rightarrow \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^2 \rightarrow \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/2][D_4]) \rightarrow 0, \qquad (2)$$

$$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/2)^2 \rightarrow \mathbb{Z}/4 \oplus \mathbb{Z}/2 \rightarrow \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/2][S_4]) \rightarrow 0. \qquad (3)$$

Next, we study the image of $\varphi : K_1(\mathbb{Z}G) \oplus K_1(\mathbb{Z}G) \rightarrow K_1(\mathbb{F}_2[G])$. Let us consider first the case $G = D_4$ for which $\mathrm{im}(\varphi) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$. This can be seen as follows: First $\mathrm{im}(\varphi) = \mathrm{im}(\psi)$ where $\psi : K_1(\mathbb{Z}[D_4]) \longrightarrow K_1(\mathbb{F}_2[D_4])$ is

induced by the canonical ring homomorphism $\mathbb{Z} \to \mathbb{F}_2$. Note the $K_1(\mathbb{Z})$ is a direct summand of $K_1(\mathbb{Z}[D_4])$ and isomorphic to $\mathbb{Z}/2$; but this summand goes to zero in $K_1(\mathbb{F}_2[D_4])$ since it factors through the following commutative square

$$\mathbb{Z}/2 = K_1(\mathbb{Z}) \longrightarrow K_1(\mathbb{F}_2) = 0$$
$$\downarrow \qquad\qquad\qquad \downarrow$$
$$K_1(\mathbb{Z}[D_4]) \longrightarrow K_1(\mathbb{F}_2[D_4]).$$

Since $K_1(\mathbb{Z}[D_4]) = (\mathbb{Z}/2)^3$, then $\dim_{\mathbb{F}_2}(\mathrm{im}(\psi)) \leqslant 2$. But from the exact sequence given in (2), $\dim_{\mathbb{F}_2}(\mathrm{im}(\psi)) \geqslant 2$, thus $\mathrm{im}(\varphi) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and it follows that $\mathrm{coker}(\varphi) \cong \mathbb{Z}/4$.

Next, for $G = S_4$ a nearly identical argument shows that the image of the map $\varphi : K_1(\mathbb{Z}[S_4]) \oplus K_1(\mathbb{Z}[S_4]) \to K_1(\mathbb{F}_2[S_4])$ is isomorphic to $\mathbb{Z}/2$ and $\mathrm{coker}(\varphi) \cong \mathbb{Z}/4$.

Hence after working through the exact sequences given in (2) and (3), we have that $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/2][D_4]) = \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/2][S_4]) = \mathbb{Z}/4$.

It follows that

$$E^2_{0,0} = H_0(\Delta^3 ; \{\tilde{K}_0(\mathbb{Z}F_\sigma)\}) = \mathbb{Z}/4 \oplus \mathbb{Z}/4.$$

$q = 1$. Oliver in [25] has shown that $Wh(F) = 0$ when F is one of the groups that occur as stabilizers of the 2-cells and the 1-cells. So it follows that $E^2_{p,1} = 0$ for $p = 1, 2$. As before for $p = 0$, the complex may have non-zero terms. The resulting homology groups are

$$E^2_{0,1} = H_0(\Delta^3 ; \{Wh(F_\sigma)\}) = Wh(D_6) \oplus Wh(\mathbb{Z}/2 \times D_4) \oplus Wh(\mathbb{Z}/2 \times S_4).$$

To calculate: $Wh(D_6)$, $Wh(\mathbb{Z}/2 \times D_4)$, and $Wh(\mathbb{Z}/2 \times S_4)$, we use the following formula given in [4]: $Wh(F) = \mathbb{Z}^y \oplus SK_1(\mathbb{Z}G)$, where y is the number of irreducible real representations of G minus the number of irreducible rational representations of G. When $G = D_6$, $\mathbb{Z}/2 \times D_4$ or $\mathbb{Z}/2 \times S_4$, y is zero, and Oliver in [25] proves that SK_1 vanishes for all finite dihedral groups. Oliver also shows in [25, Example 9.9] that if $|G|=16$, then

$$SK_1(\mathbb{Z}G) = \begin{cases} 1 & \text{if } G^{ab} \cong (C_2)^2 \text{ or } (C_2)^3, \\ \mathbb{Z}/2 & \text{if } G^{ab} \cong C_4 \times C_2. \end{cases}$$

In particular for $G = \mathbb{Z}/2 \times D_4$, $G^{ab} \cong (C_2)^3$, and we have $SK_1(\mathbb{Z}[\mathbb{Z}/2 \times D_4]) = 1$.

For the remaining case of $SK_1(\mathbb{Z}[\mathbb{Z}/2 \times S_4])$, we use the following formula due to Oliver [25, Theorem 3.9].

Let G be a finite group of order n. For each prime p, let $\hat{\mathbb{Z}}_p G$ and $\hat{\mathbb{Q}}_p G$ denote the p-adic completions of $\mathbb{Z}G$ and $\mathbb{Q}G$, and set $SK_1(\hat{\mathbb{Z}}_p G) = \ker\{K_1(\hat{\mathbb{Z}}_p G) \to K_1(\hat{\mathbb{Q}}_p G)\}$. Then set

$$CL_1(\mathbb{Z}G) = \ker\{SK_1(\hat{\mathbb{Z}}G) \xrightarrow{l} \bigoplus_p SK_1(\hat{\mathbb{Z}}_p G)\}.$$

The sum $\bigoplus_p SK_1(\hat{\mathbb{Z}}_p G)$ is, in fact, a finite sum, $SK_1(\hat{\mathbb{Z}}_p G) = 1$ whenever $p \nmid |G|$, and the localization homomorphism l is onto. In particular $SK_1(\mathbb{Z}G)$ sits in an extension

$$1 \to CL_1(\mathbb{Z}G) \to SK_1(\mathbb{Z}G) \to \bigoplus_{p|n} SK_1(\hat{\mathbb{Z}}_p G) \to 1.$$

Wall [35, Theorem 2.5] shows that $SK_1(\hat{\mathbb{Z}}_p G)$ is a p-group for any prime p and any finite group G, and $SK_1(\hat{\mathbb{Z}}_p G) = 1$ if the p-Sylow subgroup of G is abelian. Also Oliver [25, Proposition 12.7] shows that $SK_1(\hat{\mathbb{Z}}_p G) = 1$ if the p-Sylow subgroup of G has a normal abelian subgroup with cyclic quotient. In particular for $G = \mathbb{Z}/2 \times S_4$, the 3-Sylow subgroup of G is of type C_3, and the 2-Sylow subgroup of G is of type $C_2 \times D_4$. Then it follows that both $SK_1(\hat{\mathbb{Z}}_3 G)$ and $SK_1(\hat{\mathbb{Z}}_2 G)$ vanish, and we conclude $SK_1(\mathbb{Z}G) \cong CL_1(\mathbb{Z}G)$. Since the group algebra $\mathbb{R}[\mathbb{Z}/2 \times S_4]$ splits as $\mathbb{R}^4 \times (M_2(\mathbb{R}))^2 \times (M_3(\mathbb{R}))^4$, then by [25, Theorem 5.4] $CL_1(\mathbb{Z}G) = 1$, it follows that $SK_1(\mathbb{Z}G)$ vanishes. Hence the whole complex consists of zero terms, and $E_{p,q}^2 = 0$ for $q = 1$.

Thus the spectral sequence collapses at E^2, completing our computations of the algebraic K-groups $K_n(\mathbb{Z}\Gamma_3)$ for $n < 2$. $\square$

Acknowledgements

References

1. Anderson, D. R. and Hsiang, W. C.: The functors K_{-i} and pseudo-isotopies of polyhedra, *Ann. of Math.* **105** (1977), 210–223.
2. Bahls, P. and Mihalik, M.: Centralizers of parabolic subgroups of even Coxeter groups, to appear.
3. Bass, H.: The Dirichlet unit theorem, induced characters, and Whitehead groups of finite groups, *Topology* **4** (1965), 391–410.
4. Bass, H.: *Algebraic K-theory*. W. A. Benjamin, New York, 1968.

5. Berkove, E., Farrell, F. T., Pineda, D. J. and Pearson, K.: The Farrell-Jones isomorphism conjecture for finite covolume hyperbolic actions and the algebraic K-theory of Bianchi groups, *Trans. Amer. Math. Soc*. **352**(12) (2000), 5689–5702.
6. Brink, B.: On centralizers of reflections in Coxeter groups, *Bull. London Math. Soc*. **28**(5) (1996), 465–470.
7. Brown, K. S.: *Cohomology of groups*, Springer-Verlang, New York, 1982.
8. Carter, D.: Lower K-theory of finite groups, *Commun. Algebra* **8**(20) (1980), 1927–1937.
9. Carter, D.: Localization in lower algebraic K-theory, *Commun. Algebra* **8**(7) (1980), 603–622.
10. Chein, M.: Recherche des graphes des matrices de Coxeter hyperboliques d'ordre $\leqslant 10$, (French) *Rev. Française Informat. Recherche Opérationnelle Ser. R-3* **3** (1969), 3-16.
11. Coxeter, H. S. M.: *Regular Polytopes,* Dover, 3rd ed., New York, 1973.
12. Coxeter, H. S. M. and Whitrow, G. J.: World-structure and non-Euclidean honeycombs, *Proc. Roy. Soc. London. Ser. A*. **201** (1950), 417–437.
13. Curtis, C. and Reiner, I.: *Methods of Representation Theory*, vol. I, Wiley, New York, 1981.
14. Davis, M. W.: Some aspherical manifolds, *Duke Math*. J. **55**(1) (1987), 105–139.
15. Davis, J. F. and Lück, W.: Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory, *K-theory* **15**(3) (1998), 201–252.
16. Farrell, F. T. and Jones, L.: Isomorphism conjectures in algebraic K-theory, *J. Amer. Math. Soc*. **6** (2) (1993), 249–297.
17. Farrell, F. T. and Jones, L.: The lower algebraic K-theory of virtually infinite cyclic groups, *K-theory* **9** (1995), 13–30.
18. Keating, M. E.: Whitehead groups of dihedral 2-groups, In: *Algebraic K-theory, Part I (Oberwolfach, 1980)*, Lecture Notes in Math. 966, Springer, Berlin, 1982, pp. 122–127.
19. Koszul, J. L.: Lectures in hyperbolic Coxeter groups, In: Ochiai, T. (ed.), Univ. of Notre Dame, Notre Dame, 1968.
20. Lannér, F.: On complexes with transitive groups of automorphisms, *Comm. Sém., Math. Univ. Lund* [Medd. Lunds Univ. Mat. Sem.] **11** (1950).
21. Lück, W. and Stamm, R.: Computations of K- and L-theory of cocompact planar groups, *K-theory*, **21**(3)(2000), 249–292.
22. Magurn, B.: Explicit K_1 of some modular groups rings, in preparation.
23. Milnor, J.: *Introduction to algebraic K-theory*, Princeton University. Press, Princeton, 1971.
24. Munkholm, H. J. and Prassidis, S.: On the vanishing of certain K-theory Nil-groups, In: *Cohomological Methods in Homotopy Theory (Bellaterra, 1998)*, Progr. Math., Birkhäuser, Basel, **196** (2001), pp. 307-321.
25. Oliver, R.: *Whitehead Groups of Finite Groups*, London Math. Soc. Lecture Notes Ser. 132, Cambridge University Press, Cambridge, 1989.
26. Pearson, K.: Algebraic K-theory of two dimensional crystallographic groups, *K-theory* **14**(3) (1998), 265–280.
27. Pedersen, E. K. and Weibel, C.: A non-connective delooping of algebraic K-theory, In: *Algebraic and Geometric Topology Proceedings, (Rutgers, 1983)*, Lecture Notes in Math. vol. 1126, Springer-Verlag, New York, 1985, pp. 166–181.
28. Quinn, F.: Ends of maps II, *Invent. Math*. **68** (1982), 353–424.
29. Ratcliffe, J. G.: *Foundations of Hyperbolic Manifolds*, Graduate Texts in Mathematics, 149. Springer-Verlag, New York, 1994.

30. Reiner, I.: *Class groups and Picard groups of group rings and orders*, Conference Board of the Mathematical Sciences, Regional Conference Series in Mathematics, No. 26. American Mathematical Society, Providence, R. I., 1976.

31. Rosenberg, J.: *Algebraic K-theory and its Applications*, Graduate Texts in Mathematics 147, Springer-Verlag, New York, 1994.

32. Serre, J. P.: *Trees*, Springer-Verlag, Berlin, New York, 1980.

33. Upadhyay, S.: Controlled algebraic K-theory of integral group ring of $SL(3, \mathbb{Z})$, *K-theory*, **10**(4) (1996), 413–418.

34. Vinberg, E. B.: The groups of units of certain quadratic forms, (Russian) *Mat. Sb.* (N.S.) **87**(129) (1972), 18–36.

35. Wall, C. T. C.: Norms of units in group rings, *Proc. London Math. Soc.* **29**(3) (1974), 593–632.